Discipline Course-I
Semester -I
Paper: Mathematical PhysicsI IA
Lesson: First Order Homogeneous Differential Equations
Lesson Developer: Sushil Kumar Singh
College/Department: SGTB Khalsa College, University of Delhi

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Chapter 4: First Order Homogeneous Differential Equations

Learning Objectives

The student will be able to learn here

- ***what are Homogeneous Functions?***
- ***which first order differential equation will classify as Homogeneous Differential Equation?***
- ***how to reduce a Homogeneous Equation to a Variable Separable Form and find it's solution?***

First Order Homogeneous Differential Equations

4.1 Homogeneous Equations

Consider the nth order linear differential equation ***(LDE)*** of the form (1)

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \tag{1}$$

where $g(x)$ & the coefficients $a_i(x)$ $(i = 0,1,2,\ldots,n)$ either depend only on the variable x or are constants. They do not depend on y or on its derivatives. If $g(x) \neq 0$ the LDE is known as Non-Homogeneous but if $g(x) = 0$ then the LDE is known as Homogeneous.

However, in the context of *First Order DE* (which may not be Linear) written in the standard form

$$y' = f(x,y)$$

the equation is called *Homogeneous Equation* if having replaced $x \to tx$ & $y \to ty$ we find that

$$f(tx,ty) = f(x,y)$$

for every real number t.
Alternatively, we define a *Homogeneous Function* $g(x,y)$ of degree n as

$$g(tx,ty) = t^n g(x,y)$$

Now the *First Order DE* written in the differential form

$$M(x,y)dx + N(x,y)dy = 0$$

is called *Homogeneous Equation* if having replaced $x \to tx$ & $y \to ty$ we find that

$$M(tx,ty) = t^m M(x,y) \;\&\; N(tx,ty) = t^n N(x,y)$$

for every real number t with $n = m$. If this is the case, then we can write the equation as

$$y' = \frac{M(x,y)}{\{-N(x,y)\}}$$

and can cross check that

$$f(tx,ty) = \frac{M(tx,ty)}{\{-N(tx,ty)\}} = \frac{t^n M(x,y)}{\{-t^n N(x,y)\}} == \frac{M(x,y)}{\{-N(x,y)\}} = f(x,y)$$

We then conclude that the *First Order DE* $M(x,y)dx + N(x,y)dy = 0$ is a *Homogeneous Equation* if both $M(x,y)$ & $N(x,y)$are *Homogeneous Functions* of the same degree.
We like to note here that the *Homogeneous Functions* have the following properties;

- $M(x,xv) = M(1,v)$ & $N(x,xv) = N(1,v)$ where x serves as the real parameter
- $M(yv,y) = M(v,1)$ & $N(yv,y) = N(v,1)$ where y serves as the real parameter
- $f(x,xv) = f(1,v)$ where x serves as the real parameter and lastly

- $f(yv, y) = f(v, 1)$ where y serves as the real parameter

Examples of Homogeneous First Order DE:

Example 4.1.1 Determine if the following *First Order DE* is Homogeneous

$$\left\{y^2 + x^2 \sin\left(\frac{y}{x}\right)\right\} y' = 2xye^{\left(\frac{y}{x}\right)}$$

Solution: First we write the DE in the Standard Form

$$y' = \frac{2xye^{\left(\frac{y}{x}\right)}}{\left\{y^2 + x^2 \sin\left(\frac{y}{x}\right)\right\}}$$

so that we get

$$f(x, y) = \frac{2xye^{\left(\frac{y}{x}\right)}}{\left\{y^2 + x^2 \sin\left(\frac{y}{x}\right)\right\}}$$

Now changing the variable as $x \to tx \;\&\; y \to ty$ we find

$$f(tx, ty) = \frac{2(tx)(ty)e^{\left(\frac{ty}{tx}\right)}}{\left\{(ty)^2 + (tx)^2 \sin\left(\frac{ty}{tx}\right)\right\}} = \frac{t^2 \times 2xye^{\left(\frac{y}{x}\right)}}{t^2 \times \left\{y^2 + x^2 \sin\left(\frac{y}{x}\right)\right\}} = f(x, y)$$

which shows that the given equation is a *Homogeneous Equation* since $f(tx, ty) = f(x, y)$ for every real number t.

Example 4.1.2 Determine if the following *First Order DE* is Homogeneous

$$y' = \frac{xy}{\{y^2 + x^3\}}$$

Solution: The *First Order DE* is already in the Standard Form $y' = f(x, y)$ so

$$f(x, y) = \frac{xy}{\{y^2 + x^3\}}$$

Now changing the variable as $x \to tx \;\&\; y \to ty$ we find

$$f(tx, ty) = \frac{(tx)(ty)}{\{(ty)^2 + (tx)^3\}} = \frac{t^2 \times xy}{t^2 \times \{y^2 + tx^3\}} \neq f(x, y)$$

which shows that the given equation is *Not* a *Homogeneous Equation*.

4.2 Reduction of Homogeneous Equation to Variable Separable Form

The Homogeneous DE

$$y' = f(x, y)$$

having the property $f(tx, ty) = f(x, y)$ can be transformed into a Separable DE by either :

4.2.1 Case where we try putting $y = xv$

Putting $y = xv$ we get

$$y' = xv' + v$$

Substituting this back in the DE leads to

$$xv' + v = f(x, vx)$$

and since f is homogeneous of degree n we find $f(x, vx) = f(1, v)$. Thus,

$$xv' + v = f(1, v)$$
$$xv' = f(1, v) - v$$
$$\frac{dv}{\{f(1, v) - v\}} = \frac{dx}{x}$$

Having successfully separated the variables we can now integrate the equation

$$\int \frac{dv}{\{f(1, v) - v\}} = \int \frac{dx}{x}$$

$$\int \frac{dv}{\{f(1, v) - v\}} = \ln x + C$$

Examples of Reduction of Homogeneous First Order DE for the Case 3.2.1 :

Example 4.2.1.1 Solve the DE by reducing it to separable equation

$$\boldsymbol{y' = \frac{(y + 2x)}{3x}}$$

Solution: Since the *First Order DE* is already in the Standard Form $y' = f(x, y)$ and appears not be easily Separable, so first we check for its Homogeneity

$$f(tx, ty) = \frac{(ty + 2tx)}{3tx} = \frac{t \times (y + 2x)}{t \times x} = f(x, y)$$

Now we know it is homogeneous, so can change the variable to $y = vx$ then

$$xv' + v = \frac{(vx + 2x)}{3x} = \frac{(v + 2)}{3}$$

which separates the variable as

$$x\frac{dv}{dx} = \frac{(v+2)}{3} - v = \frac{(v+2-3v)}{3} = \frac{2(1-v)}{3}$$
$$\frac{dv}{(v-1)} = -\frac{2}{3}\frac{dx}{x}$$

Integration will now yield

$$\int \frac{dv}{(v-1)} = -\frac{2}{3}\int \frac{dx}{x}$$
$$\ln|(v-1)| = -\frac{2}{3}\ln x + C = C + \ln x^{-2/3}$$
$$v - 1 = x^{-2/3}e^{C}$$
$$v = 1 + Ax^{-2/3}$$

where $A = e^{C}$ and so is always positive. Substituting back $v = y/x$ we finally arrive at

$$y(x) = x\left(1 + Ax^{-2/3}\right) = x + Ax^{1/3}$$

which is an *explicit* solution for the DE.

Example 4.2.1.2 Solve the DE by reducing it to separable equation

$$\boldsymbol{y' = \frac{(2y^n + x^n)}{xy^{n-1}}}$$

for some real n.

Solution: Since the *First Order DE* is already in the Standard Form $y' = f(x, y)$ and appears not be easily Separable, so first we check for its Homogeneity

$$f(tx, ty) = \frac{[2(ty)^n + (tx)^n]}{(tx)(ty)^{n-1}} = \frac{t^n \times (2y^n + x^n)}{t^n \times xy^{n-1}} = f(x, y)$$

Now we know it is homogeneous, so to reduce it into separable form we put $y = vx$ and use $y' = xv' + v$ to get

$$xv' + v = \frac{[2(vx)^n + x^n]}{x(vx)^{n-1}} = \frac{x^n \times (2v^n + 1)}{x^n \times v^{n-1}} = \frac{(2v^n + 1)}{v^{n-1}}$$

which separates the variable as

$$x\frac{dv}{dx} = \frac{(2v^n + 1)}{v^{n-1}} - v = \frac{(2v^n + 1 - v^n)}{v^{n-1}} = \frac{(v^n + 1)}{v^{n-1}}$$
$$\frac{v^{n-1}dv}{(v^n + 1)} = \frac{dx}{x}$$

The General Solution is now obtained by integrating the Separable DE

$$\int \frac{v^{n-1}dv}{(v^n + 1)} = \int \frac{dx}{x}$$
$$\ln|(v^n + 1)| = n\ln x + C = C + \ln x^n$$
$$v^n + 1 = x^n e^{C}$$

$$\frac{y^n}{x^n} = -1 + Ax^n$$

where $A = e^C$ and so is always positive. Substituting back $v = y/x$ we finally arrive at

$$y^n = x^n(-1 + Ax^n) = -x^n + Ax^{2n}$$

which is an *explicit* solution for the DE.

4.2.2 Case where we try putting $x = yv$

Putting $x = yv$ we get

$$x' = yv' + v$$

Rewriting the DE as

$$\frac{dx}{dy} = \frac{1}{f(x,y)}$$

and substituting this leads to

$$yv' + v = \frac{1}{f(vy,y)}$$

But as before, since f is homogeneous of degree n we find $f(vy,y) = f(v,1)$. Thus,

$$yv' + v = \frac{1}{f(v,1)} \equiv f^{-1}(v,1)$$
$$xv' = f^{-1}(v,1) - v$$
$$\frac{dv}{\{f^{-1}(v,1) - v\}} = \frac{dx}{x}$$

Having successfully separated the variables we can now integrate the equation

$$\int \frac{dv}{\{f^{-1}(v,1) - v\}} = \int \frac{dx}{x}$$

$$\int \frac{dv}{\{f^{-1}(v,1) - v\}} = \ln x + C$$

Examples of Reduction Homogeneous First Order DE for the Case 3.2.2 :

Example 4.2.2.1 Solve the DE by reducing it to separable equation

$$\boldsymbol{y' = \frac{xye^{(x/y)^2}}{\{y^2 + y^2e^{(x/y)^2} + x^2e^{(x/y)^2}\}}}$$

Solution: Since the *First Order DE* is already in the Standard Form $y' = f(x,y)$ and appears not be easily Separable, so first we check for its Homogeneity

$$f(tx,ty)=\frac{(tx)(ty)e^{(tx/ty)^2}}{\{(ty)^2+(ty)^2e^{(tx/ty)^2}+(tx)^2e^{(tx/ty)^2}\}}=\frac{t^2\times xye^{(x/y)^2}}{t^2\times\{y^2+y^2e^{(x/y)^2}+x^2e^{(x/y)^2}\}}=f(x,y)$$

Now we know it is homogeneous, so to reduce it into separable form we change the variable by writing $x=vy$ and using $x'=yv'+v$. For this we have to rewrite the equation as (inverting the given DE)

$$x'=\frac{\{y^2+y^2e^{(x/y)^2}+x^2e^{(x/y)^2}\}}{xye^{(x/y)^2}}$$

$$yv'+v=\frac{\{y^2+y^2e^{(vy/y)^2}+(vy)^2e^{(vy/y)^2}\}}{(vy)(y)e^{(vy/y)^2}}=\frac{y^2\times\{1+e^{v^2}+v^2e^{v^2}\}}{y^2\times ve^{v^2}}=\frac{\{1+e^{v^2}+v^2e^{v^2}\}}{ve^{v^2}}$$

which separates the variable as

$$y\frac{dv}{dy}=\frac{\{1+e^{v^2}+v^2e^{v^2}\}}{ve^{v^2}}-v=\frac{\{(1+e^{v^2})+v^2e^{v^2}-v^2e^{v^2}\}}{ve^{v^2}}=\frac{\{1+e^{v^2}\}}{ve^{v^2}}$$

$$\frac{ve^{v^2}}{\{1+e^{v^2}\}}dv=\frac{dy}{y}$$

The General Solution is now obtained by integrating the Separable DE

$$\int\frac{ve^{v^2}}{\{1+e^{v^2}\}}dv=\int\frac{dy}{y}$$

$$\ln\left(1+e^{v^2}\right)=2\ln y+C=C+\ln y^2$$

$$1+e^{v^2}=y^2e^C$$

$$1+e^{v^2}=Ay^2$$

where $A=e^C$ and so is always positive. Substituting back $v=x/y$ we finally arrive at

$$1+e^{(x/y)^2}=Ay^2$$

$$y^2=B\left(1+e^{(x/y)^2}\right)$$

which is an *implicit* solution for the DE.

Example 4.2.2.2 Solve the DE by reducing it to separable equation

$$\boldsymbol{y'=\frac{(2y^n+x^n)}{xy^{n-1}}}$$

for some real n.

Solution: We have already seen that the *First Order DE* is Homogeneous, so let's try to flip the equation to get

$$x' = \frac{xy^{n-1}}{(2y^n + x^n)}$$

Now we know it is homogeneous, so to reduce it into separable form. We put $x = vy$ and use $x' = yv' + v$ to get

$$yv' + v = \frac{vyy^{n-1}}{[2y^n + (vy)^n]} = \frac{y^n \times v}{y^n \times (2 + v^n)} = \frac{v}{(2 + v^n)}$$

which separates the variable as

$$y\frac{dv}{dy} = \frac{v}{(2 + v^n)} - v = \frac{(v - 2v - v^{n+1})}{(2 + v^n)} = -\frac{v(v^n + 1)}{(2 + v^n)}$$

$$\frac{(2 + v^n)dv}{v(v^n + 1)} = -\frac{dy}{y}$$

The General Solution is now obtained by integrating the Separable DE

$$\int \left\{\frac{(2 + 2v^n - v^n)}{v(v^n + 1)}\right\} dv = -\int \frac{dy}{y}$$

$$\int \left\{\frac{2}{v} - \frac{v^{n-1}}{(v^n + 1)}\right\} dv = -\int \frac{dy}{y}$$

$$2n \ln v - \ln|(v^n + 1)| = -n \ln y + C$$

$$\frac{v^{2n}}{v^n + 1} = y^{-n}e^C$$

Putting $A = e^C$ (and so is always positive)

$$y^n = A\left(\frac{v^n + 1}{v^{2n}}\right)$$

Substituting back $v = x/y$ we finally arrive at

$$y^n = A\left(\frac{\left({}^x/_y\right)^n + 1}{\left({}^x/_y\right)^{2n}}\right)$$

$$y^n = A\left(\frac{(xy)^n + y^{2n}}{x^{2n}}\right)$$

$$y^n x^{2n} = A[(xy)^n + y^{2n}]$$

$$x^{2n} = A[x^n + y^n]$$

$$Bx^{2n} = x^n + y^n$$

$$y^n = -x^n + Bx^{2n}$$

which is an *explicit* solution for the DE and where $B = 1/A$.

4.3 Mixed bag of Examples

Example 4.3.1 Solve the DE by first checking for homogeneity

$$\boldsymbol{\alpha^2 yy' + x = 0}$$

Solution: First we write the DE in the Standard Form

$$y' = -\alpha^{-2}\frac{x}{y}$$

Its easy to check for its Homogeneity

$$f(tx,ty) = -\alpha^{-2}\frac{tx}{ty} = f(x,y)$$

Now we know it is homogeneous, so to reduce it into separable form we can put $y = vx$ and use $y' = xv' + v$. For this we have to rewrite the equation as

$$xv' + v = -\frac{\alpha^{-2}}{v}$$
$$xv' = -\frac{(\alpha^{-2} + v^2)}{v}$$

By separating the variables, we get

$$\frac{vdv}{(\alpha^{-2} + v^2)} = -\frac{dx}{x}$$

The General Solution is now obtained by integrating the Separable DE

$$\int \frac{2vdv}{(\alpha^{-2} + v^2)} = -2\int \frac{dx}{x}$$
$$\ln(v^2 + \alpha^{-2}) = -2\ln x + C = \ln x^{-2} + C$$
$$v^2 + \alpha^{-2} = e^C x^{-2}$$
$$x^2v^2 + \left(\frac{x}{\alpha}\right)^2 = e^C$$

Substituting back $v = y/x$ we finally arrive at

$$\alpha^2 y^2 + x^2 = A$$

where $e^C\alpha^2 = A$ is the arbitrary constant. This is an *implicit* solution for the DE.

We can look for the *explicit* solution of the form by solving for $y(x)$

$$y(x) = \pm\sqrt{\frac{A^2 - x^2}{\alpha^2}}$$

Example 4.3.2 Solve the DE by first checking for homogeneity

$$\boldsymbol{xy' = y + x\cos^2\frac{y}{x}}$$

Solution: First we write the DE in the Standard Form

$$y' = \frac{y + x\cos^2\frac{y}{x}}{x}$$

Its easy to check for its Homogeneity

$$f(tx,ty) = \frac{ty + tx\cos^2\frac{ty}{tx}}{tx} = \frac{t\times\left(y + x\cos^2\frac{y}{x}\right)}{t\times x} = f(x,y)$$

Now we know it is homogeneous, so to reduce it into separable form we can put $y = vx$ and use $y' = xv' + v$. For this we have to rewrite the equation as (you can use the *Thumb Rule* $y \to v \,\&\, x \to 1$ in $f(x,y)$; can be understood as $f(x,xv) = f(1,v)$)

$$xv' + v = \frac{v + \cos^2 v}{1} = v + \cos^2 v$$

By separating the variable we get

$$xv' = \cos^2 v$$
$$\frac{dv}{\cos^2 v} = \frac{dx}{x}$$

The General Solution is now obtained by integrating the Separable DE

$$\int \sec^2 v\, dv = \int \frac{dx}{x}$$
$$\tan v = \ln|x| + C$$

Substituting back $v = y/x$ we finally arrive at

$$\tan\frac{y}{x} = \ln|x| + C$$

where C is the arbitrary constant. This is an *implicit* solution for the DE.

We can look for the *explicit* solution of the form by solving for $y(x)$

$$y(x) = x\tan^{-1}\{\ln|x| + C\}$$

Example 4.3.3 Solve the DE by first checking for homogeneity

$$\mathbf{2y' = \frac{y}{x} + \frac{y^2}{x^2}}$$

Solution: The DE in the Standard Form

$$y' = f(x,y) = \frac{1}{2}\left(\frac{y}{x} + \frac{y^2}{x^2}\right)$$

It’s easy to check for its Homogeneity

$$f(tx,ty) = \frac{1}{2}\left[\frac{ty}{tx} + \frac{(ty)^2}{(tx)^2}\right] = f(x,y)$$

Now we know it is homogeneous, so to reduce it into separable form we can put $y = vx$ and use $y' = xv' + v$. For this we have to rewrite the equation as (again using the *Thumb Rule* $y \to v$ & $x \to 1$ in $f(x,y)$)

$$xv' + v = \frac{1}{2}\left(\frac{v}{1} + \frac{v^2}{1^2}\right) = \frac{1}{2}v + \frac{1}{2}v^2$$

By separating the variable we get

$$xv' = \frac{1}{2}v + \frac{1}{2}v^2 - v = \frac{1}{2}v^2 - \frac{1}{2}v = \frac{1}{2}v(v-1)$$
$$\frac{dv}{v(v-1)} = \frac{1}{2}\frac{dx}{x}$$
$$\frac{(1-v+v)dv}{v(v-1)} = \frac{1}{2}\frac{dx}{x}$$

The General Solution is now obtained by integrating the Separable DE

$$\int\left\{-\frac{dv}{v} + \frac{dv}{(v-1)}\right\} = \frac{1}{2}\int\frac{dx}{x}$$
$$-\ln v + \ln(v-1) = \frac{1}{2}\ln|x| + C$$
$$-2\ln v + 2\ln(v-1) = \ln|x| + C$$
$$\left[\frac{(v-1)}{v}\right]^2 = \left(1 - \frac{1}{v}\right)^2 = e^C x$$

Substituting back $v = y/x$ we finally arrive at

$$1 - \frac{x}{y} = \pm\sqrt{e^C x}$$
$$y = \frac{x}{(1 \pm B\sqrt{x})}$$

where $B = \sqrt{e^C}$ is the arbitrary constant. This is an *explicit* general solution for the DE.

Example 4.3.4 Solve the DE by first checking for homogeneity

$$\boldsymbol{x\sin\left(\frac{y}{x}\right)y' = y\sin\left(\frac{y}{x}\right) - x}$$

Solution: The DE in the Standard Form

$$y' = f(x,y) = \frac{\left[y\sin\left(\frac{y}{x}\right) - x\right]}{x\sin\left(\frac{y}{x}\right)}$$

It's easy to check for its Homogeneity

$$f(tx, ty) = \frac{\left[ty \sin\left(\frac{ty}{tx}\right) - tx\right]}{tx \sin\left(\frac{ty}{tx}\right)} = \frac{t \times \left[y \sin\left(\frac{y}{x}\right) - x\right]}{t \times x \sin\left(\frac{y}{x}\right)} = f(x, y)$$

Now we know it is homogeneous, so to reduce it into separable form we can put $y = vx$ and use $y' = xv' + v$. For this we have to rewrite the equation as (*Thumb Rule* $y \to v$, $x \to 1$ & $\frac{y}{x} \to v$ in $f(x, y)$)

$$xv' + v = \frac{[v \sin\ v - 1]}{\sin\ v}$$

By separating the variable we get

$$xv' = \frac{[v \sin\ v - 1]}{\sin\ v} - v$$

$$xv' = \left[v - \frac{1}{\sin\ v}\right] - v = -\frac{1}{\sin\ v}$$

$$-\sin\ v\, dv = \frac{dx}{x}$$

The General Solution is now obtained by integrating the Separable DE

$$\int -\sin\ v\, dv = \int \frac{dx}{x}$$

$$\cos\ v = \ln|x| + C$$

Substituting back $v = y/x$ we finally arrive at

$$\cos\left(\frac{y}{x}\right) = \ln|x| + C$$

$$y = x\ \cos^{-1}(\ln|x| + C)$$

where C is the arbitrary constant. This is an *explicit* general solution for the DE.

Example 4.3.5 Solve the DE by first checking for homogeneity

$$\boldsymbol{(x - y - 2) + (x - 2y - 3)y' = 0}$$

Solution: The DE in the Standard Form

$$y' = f(x, y) = -\frac{(x - y - 2)}{(x - 2y - 3)}$$

Although this isn't Homogeneous, we can make a change of variable $X = x - h$ & $Y = y - k$ and the DE reduces to

$$y' = f(x, y) = -\frac{[(X + h) - (Y + k) - 2]}{[(X + h) - 2(Y + k) - 3]} = -\frac{[X - Y + (h - k - 2)]}{[X - 2Y + (h - 2k - 3)]}$$

The equation becomes Homogeneous if we can choose h & k such that

$$h - k - 2 = 0$$

$$h - 2k - 3 = 0$$

This requires

$$h = 1 \,\&\, k = -1$$

and the DE becomes

$$Y' = f(X,Y) = -\frac{(X-Y)}{(X-2Y)}$$

It's easy to check for its Homogeneity

$$f(tX,tY) == -\frac{(tX-tY)}{(tX-2tY)} = f(X,Y)$$

Now we know it is homogeneous, so to reduce it into separable form we can put $Y = vX$ and use $Y' = Xv' + v$. For this we have to rewrite the equation as (*Thumb Rule* $Y \to v\,, X \to 1 \,\&\, \frac{Y}{X} \to v$ in $f(X,Y)$)

$$Xv' + v = -\frac{(1-v)}{(1-2v)}$$

By separating the variable we get

$$Xv' = -\frac{(1-v)}{(1-2v)} - v = -\left\{\frac{(1-v)+v(1-2v)}{(1-2v)}\right\} = \frac{(2v^2-1)}{(2v-1)}$$

$$\frac{(2v-1)dv}{(2v^2-1)} = \frac{dX}{X}$$

$$\frac{2vdv}{(2v^2-1)} - \frac{dv}{(2v^2-1)} = \frac{dX}{X}$$

$$\frac{2vdv}{(2v^2-1)} - \frac{1}{2}\left\{\frac{dv}{(\sqrt{2}v-1)} - \frac{dv}{(\sqrt{2}v+1)}\right\} = \frac{dX}{X}$$

The General Solution is now obtained by integrating the Separable DE

$$\int\left[\frac{2vdv}{(2v^2-1)} - \frac{1}{2}\left\{\frac{dv}{(\sqrt{2}v-1)} - \frac{dv}{(\sqrt{2}v+1)}\right\}\right] = \int\frac{dX}{X}$$

$$\frac{1}{2}\int\left\{\frac{d(2v^2)}{(2v^2-1)} - \frac{dv}{(\sqrt{2}v-1)} + \frac{dv}{(\sqrt{2}v+1)}\right\} = \int\frac{dX}{X}$$

$$\ln|2v^2-1| - \frac{1}{\sqrt{2}}\ln\left|\sqrt{2}v-1\right| + \frac{1}{\sqrt{2}}\ln\left|\sqrt{2}v+1\right| = 2\ln|X| + C$$

$$\ln|2v^2-1| + \frac{1}{\sqrt{2}}\ln\frac{\left|\sqrt{2}v+1\right|}{\left|\sqrt{2}v-1\right|} = 2\ln|X| + C$$

Substituting back $v = Y/X$ we finally arrive at

$$\ln\left|2\left(\frac{Y}{X}\right)^2 - 1\right| + \frac{1}{\sqrt{2}}\ln\frac{\left|\sqrt{2}\left(\frac{Y}{X}\right)+1\right|}{\left|\sqrt{2}\left(\frac{Y}{X}\right)-1\right|} = 2\ln|X| + C$$

Putting back $X \,\&\, Y$

$$\ln\left|2\left(\frac{y+1}{x-1}\right)^2 - 1\right| + \frac{1}{\sqrt{2}}\ln\frac{\left|\sqrt{2}\left(\frac{y+1}{x-1}\right)+1\right|}{\left|\sqrt{2}\left(\frac{y+1}{x-1}\right)-1\right|} = 2\ln|x-1| + C$$

where C is the arbitrary constant. This is an *implicit* general solution for the DE.

4.4 Some Rules to End

***Rule 1*. Prove that if $y' = f(x,y)$ is homogeneous then the DE can be written as $y' = g(y/x)$ where $g(y/x)$ is dependent only on the quotient y/x.**

Solution: Now for DE to be homogeneous we know that

$$f(tx,ty) = f(x,y) \quad \forall \quad values\ of\ t$$

and that

$$f(x,y) = \frac{M(x,y)}{\{-N(x,y)\}}$$

where both $M(x,y)$ & $N(x,y)$are *Homogeneous Functions* of the same degree n

$$M(tx,ty) = t^n M(x,y) \,\&\, N(tx,ty) = t^n N(x,y)$$

Since it holds for all values of t, we can choose it to in particular assume the value $t = 1/x$. So, we get

$$M\left(1,\frac{1}{x}y\right) = \left(\frac{1}{x}\right)^n M(x,y) \,\&\, N\left(1,\frac{1}{x}y\right) = \left(\frac{1}{x}\right)^n N(x,y)$$

$$f(x,y) = \frac{M(x,y)}{\{-N(x,y)\}} = \frac{M\left(1,\frac{y}{x}\right)}{\left\{-N\left(1,\frac{y}{x}\right)\right\}} = f\left(1,\frac{y}{x}\right)$$

Now we can define

$$g\left(\frac{y}{x}\right) \equiv f\left(1,\frac{y}{x}\right)$$

and the *DE* reduces to

$$y' = f(x,y) = f\left(1,\frac{y}{x}\right) = g\left(\frac{y}{x}\right)$$

Note that if we now write $v = y/x$ then

$$\boldsymbol{xv' + v = g(v) = f(1,v)}$$

which is the short cut we have used earlier.

Summary

Homogeneous Equations

- In the context of *First Order DE* (which may not be Linear) written in the standard form $y' = f(x, y)$ the equation is called *Homogeneous Equation* if having replaced $x \to tx$ & $y \to ty$ we find that $f(tx, ty) = f(x, y)$ for every real number t.
- The *First Order DE* $M(x, y)dx + N(x, y)dy = 0$ is a *Homogeneous Equation* if both $M(x, y)$ & $N(x, y)$are *Homogeneous Functions* of the same degree.
- We like to note here that the *Homogeneous Functions* have the following properties;

  $M(x, xv) = M(1, v)$ & $N(x, xv) = N(1, v)$ where x serves as the real parameter
  $M(yv, y) = M(v, 1)$ & $N(yv, y) = N(v, 1)$ where y serves as the real parameter
  $f(x, xv) = f(1, v)$ where x serves as the real parameter and lastly
  $f(yv, y) = f(v, 1)$ where y serves as the real parameter

Reduction of Homogeneous Equation to Variable Separable Form

- The Homogeneous DE $y' = f(x, y)$ can be transformed into a Separable DE by either : putting $y = xv$ & $y' = xv' + v$ or putting $\boldsymbol{x = yv}$ & $\boldsymbol{x' = yv' + v}$
- If $\boldsymbol{y' = f(x, y)}$ is homogeneous then the DE can be written as $xv' + v = g(v) = f(1, v)$ where $v = y/x$.

Bibliography/ References / Glossary

1. Advanced Engineering Mathematics by Erwin Kreysig
2. Advanced Engineering Mathematics by Michael D. Greenberg
3. Schaum's Outline: Theory and Problems of Advanced Calculus by Murray R. Spiegel
4. Mathematical Methods in Physical Sciences by Mary L. Boas
5. Calculus & Analytic Geometry by Fobes & Smyth
6. Essential Mathematical Methods by K.F. Riley & M.P. Hobson
7. Schaum's Outline: Theory and Problems of Differential Equations by Richard Bronson
8. Schaum's Outline: Theory and Problems of Differential Equations by Frank Ayres
9. Introductory Course in Differential Equations by Daniel A. Murray
10. Differential Equations by N.M. Kapoor
11. Higher Engineering Mathematics by B S Grewal
12. A Treatise on Differential Equations by A. R. Forsyth