Discipline Course-I
Semester -I
Paper: Mathematical PhysicsI IA
Lesson: First Order Separable Differential Equations

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Table of Contents

Chapter 3: First Order Separable Differential Equations

Learning Objectives

After reading this chapter the student will be able to learn

- ***How to represent any general first order Differential equation in a Standard?***
- ***How to represent any general first order Differential equation in a Differential Form which at times is more useful than the standard form? This lesson is dedicated to a special form of differential equation called Separable Equations. There is a huge class of problems in physics where the separable equations come in handy.***

First Order Separable Differential Equations

Differential equations of first order are of special value in science, engineering and social sciences. They are a link between mathematics and science. They help us understand curves & trajectories; they allow us to model growth and decay of life forms, radioactivity, economy and many physical phenomena. Its knowledge is a prerequisite to understand the basics of physical sciences. Their abundance helps in understanding almost all forms of Natural Sciences.

3.1 Standard and Differential Form

The general first order differential equation is given by

$$F(x, y, y') = 0 \tag{1}$$

where x and y are independent and dependent variables respectively and y' is of ***degree 1***.
The equation (1) can be solved algebraically for y' and re-expressed in the **Standard Form** as

$$y' = f(x, y) \tag{2}$$

It is to be noted that not all equations can be written in the standard form. However, the DE in Standard Form $y' = f(x, y)$ can be written as a quotient of two other functions $M(x, y)$ and $-N(x, y)$ such that

$$y' = \frac{M(x,y)}{\{-N(x,y)\}} \tag{3}$$

and since $y' = dy/dx$ this can be re-expressed in the **Differential Form** as

$$M(x, y)dx + N(x, y)dy = 0 \tag{4}$$

Examples of Standard and Differential Form

Example 3.1.1 Write the DE

$$\boldsymbol{e^x y' + e^{2x} y = sinx}$$

in standard form.

Solution: We need the form $y' = f(x, y)$ which we obtain by rearranging as

$$e^x y' = sinx - e^{2x} y$$
$$y' = e^{-x} sinx - e^x y$$

Example 3.1.2 Write the DE

$$\boldsymbol{(y' + y)^3 = sin\left(\frac{y'}{x}\right)}$$

in standard form.

Solution: This DE cannot be solved algebraically for y' and cannot be written in the standard form.

Example 3.1.3 Write the Differential form of DE

$$\boldsymbol{y(yy'-1)=x}$$

Solution:

$$y^2y'-y=x$$
$$y^2y'=x+y$$
$$y'=\frac{x+y}{y^2}$$

is the standard form.

We choose $M(x,y)$ and $-N(x,y)$ having any form such that $y'=\frac{x+y}{y^2}$ and so there are infinitely many different differential forms associated with equation (1).

Let's try to understand how to construct differential equations in the following examples;

Example 3.1.4 In a radioactive decay an element Y decays at a rate proportional to the amount y of the element Y present. Find the DE which represents such a solution and plot the family of curves obtained from the solution.

Solution: It has been given that the element Y decays at the rate

$$-\frac{dy}{dt}=\alpha y$$

By separating the variable we get

$$\frac{dy}{y}=-\alpha dt$$

We can now integrate the equation

$$\int\frac{dy}{y}=-\alpha\int dt$$
$$\ln y=-\alpha(t+C)$$

and write it in the form in the *explicit*

$$y(t)=Ae^{-\alpha t}$$

where $A=e^{-\alpha C}$ is the arbitrary constant.

But what does A represent? We can see that

$$A=y(0)$$

so A represent the initial amount (number) of the radioactive element Y. And what α does represent? We can see that at the time $t=T_{1/2}$ when the radioactive element has reduced to $y=\frac{y(0)}{2}=\frac{A}{2}$

$$y(T_{1/2})=Ae^{-\alpha T_{1/2}}$$
$$\frac{A}{2}=Ae^{-\alpha T_{1/2}}$$
$$e^{\alpha T_{1/2}}=2$$

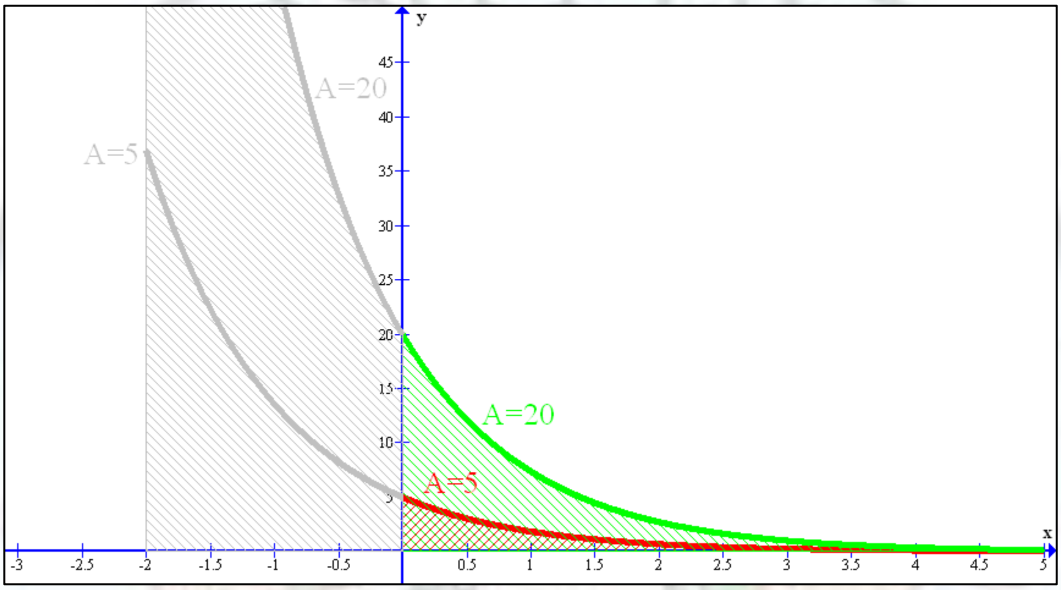
$$\alpha = \frac{\ln 2}{T_{1/2}}$$

So α is the Decay Constant inverse of which is proportional to the Half Life of the radioactive element.

We can also write the solution as

$$y(t) = y(0)e^{-\alpha t} = y(0)e^{-\ln 2 \frac{t}{T_{\frac{1}{2}}}} = y(0)\left(\frac{1}{2}\right)^{t/T_{1/2}}$$

Since y represents the amount of the radioactive element, it has to be a positive quantity which restricts our choice of arbitrary constant A to positive values only. Let $\alpha = 1$ then $y(t) = Ae^{-t}$ and some representative curves would be



Example 3.1.5 The population of a city increases at a rate proportional to the present population N and the difference between a maximum population of N_0 & the present population$N$. Find the DE which represents such a solution and plot the family of curves obtained from the solution.

Solution: It has been given that

$$\frac{dN}{dt} = \alpha N(N_0 - N)$$

By separating the variable we get

$$\frac{dN}{N(N_0 - N)} = \alpha dt$$

We can now integrate the equation

$$\int \frac{N_0 dN}{N(N_0 - N)} = \alpha N_0 \int dt$$

$$\int \frac{(N_0 - N + N)dN}{N(N_0 - N)} = \alpha N_0 \int dt$$

$$\int \left[\frac{1}{N} + \frac{1}{(N_0 - N)}\right] dN = \alpha N_0 \int dt$$

$$\int \left[\frac{1}{N} - \frac{1}{(N - N_0)}\right] dN = \alpha N_0 \int dt$$
$$\ln N - \ln(N - N_0) = \alpha N_0 (t + C)$$
$$\ln \frac{N}{(N - N_0)} = \alpha N_0 (t + C)$$
$$\frac{N}{(N - N_0)} = Ae^{\alpha N_0 t}$$
$$1 - \frac{N_0}{N} = \frac{1}{A} e^{-\alpha N_0 t}$$
$$1 - Be^{-\alpha N_0 t} = \frac{N_0}{N}$$

where $A = e^{\alpha N_0 C}$ and write it in the explicit form

$$N = \frac{N_0}{(1 - Be^{-\alpha N_0 t})}$$

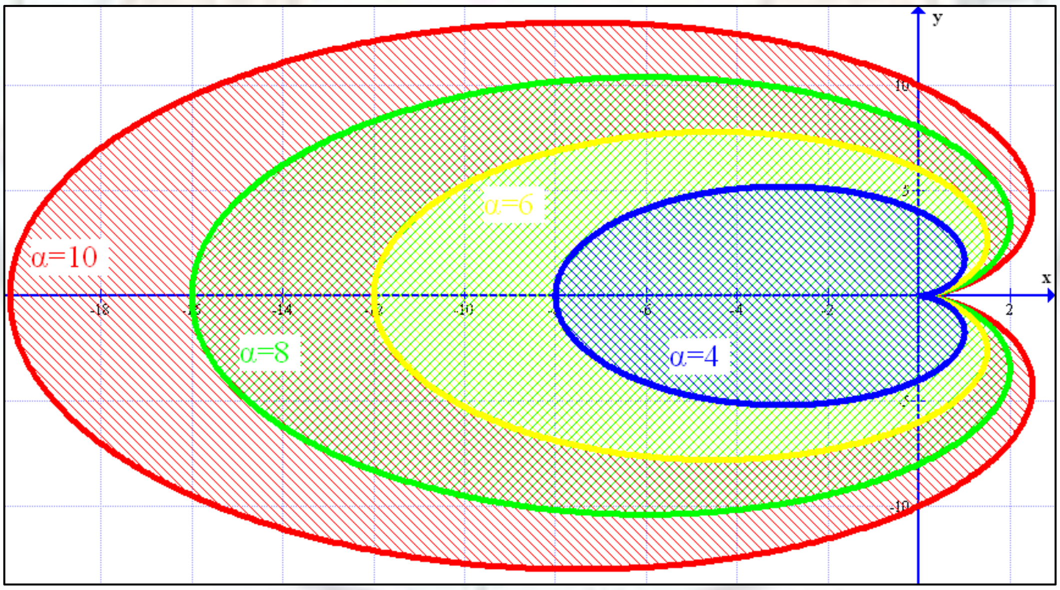
where $B = \frac{1}{A}$ is the arbitrary constant.

Example 3.1.6 Find the DE which represent the *cardioids*.

$$\boldsymbol{r = \alpha(1 - \cos\theta)}$$

Plot the family of curves obtained from the solution.

Solution: Let us first plot few *cardioids* by taking the parameter $\alpha = 10, 8, 6 \; \& \; 4$.



We can see it's likely that every point on the (r, θ) plane has a *unique* cardioid passing through it having a particular α. We will find that this is the general *property of equation of First Order First Degree*.

Taking the first derivative with respect to angle θ,

$$\frac{dr}{d\theta} = -\alpha \sin\theta$$

Using $\alpha = r/(1 - \cos\theta)$ we get

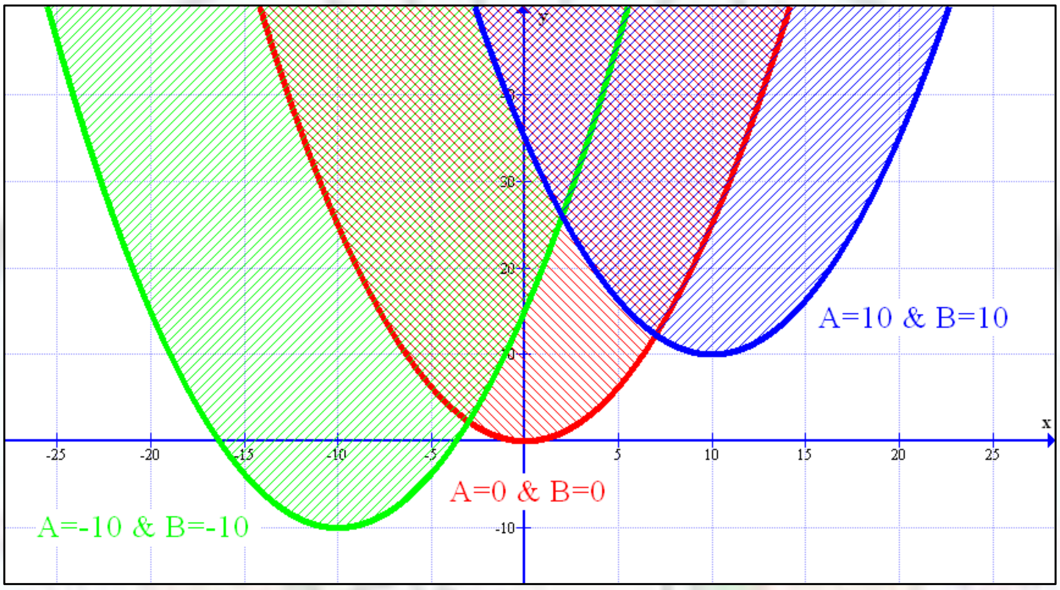
$$\frac{dr}{d\theta} = -\frac{r}{(1-\cos\theta)}\sin\theta$$
$$(1-\cos\theta)dr + r\sin\theta\, d\theta = 0$$

which represents the differential equation for *cardioids*.

Example 3.1.7 Plot the family of curves which represent the *parabolas* with latus rectum $4a$ and axes parallel to the y-axis. Also obtain the DE

$$(\boldsymbol{x} - \boldsymbol{A})^2 = 4\boldsymbol{a}(\boldsymbol{y} - \boldsymbol{B})$$

Solution: Let's plot few *parabolae* by taking the parameter $(A, B) = (0,0), (-10,-10)$ & $(10,10)$.



We can see it's likely that every point on the (x, y) plane has a *unique* parabola passing through it having a particular(A, B) and having particular value of slope. We will find that this is the general *property of equation of Linear Second Order*.
Taking the first derivative with respect to x,

$$2(x - A) = 4a\frac{dy}{dx}$$

Taking the second derivative with respect to x,

$$\frac{d^2y}{dx^2} = \frac{1}{2a}$$

which represents the differential equation.

3.2 Separable Equations

Consider a 1st order differential equation in differential form (4)

$$M(x, y)dx + N(x, y)dy = 0$$

Now if this DE can be written such that

$$M(x,y) = X(x) \quad : \text{a function of } x \text{ only}$$
$$N(x,y) = Y(y) \quad : \text{a function of } y \text{ only}$$

then we can represent it as

$$X(x)dx + Y(y)dy = 0 \tag{5}$$

which is a separable differential equation of the form

$$y' = \frac{X(x)}{\{-Y(y)\}}$$

We can solve this equation by writing

$$X(x)dx = -Y(y)dy$$

and integrating both sides

$$\int X(x)dx = -\int Y(y)dy + C$$

These two integrals exist if $X(x)$ and $Y(y)$ are continuous function of x over the interval.

We note that $y' = \frac{X(x)}{\{-Y(y)\}}$ can be rewritten as

$$y' = P(x)Q(y)$$

by representing $P(x) = X(x)$ and $Q(y) = \frac{1}{\{-Y(y)\}}$ the DE is expressed as a product of two functions which are independent of each other.

Let's try to understand what a separable differential equation is and how we use this fact to arrive at a solution .Also lets understand, how we can obtain an explicit solution and what solution in particular satisfies the initial values given at some point.

Examples of Separable 1st Order DE

Example 3.2.1 Solve the DE by separating variables

$$\boldsymbol{\alpha^2 yy' + x = 0}$$

and plot the family of curves obtained from the solution.

Solution: First we write the DE in the Differential Form

$$\alpha^2 ydy + xdx = 0$$

By separating the variable we get

$$\alpha^2 ydy = -xdx$$

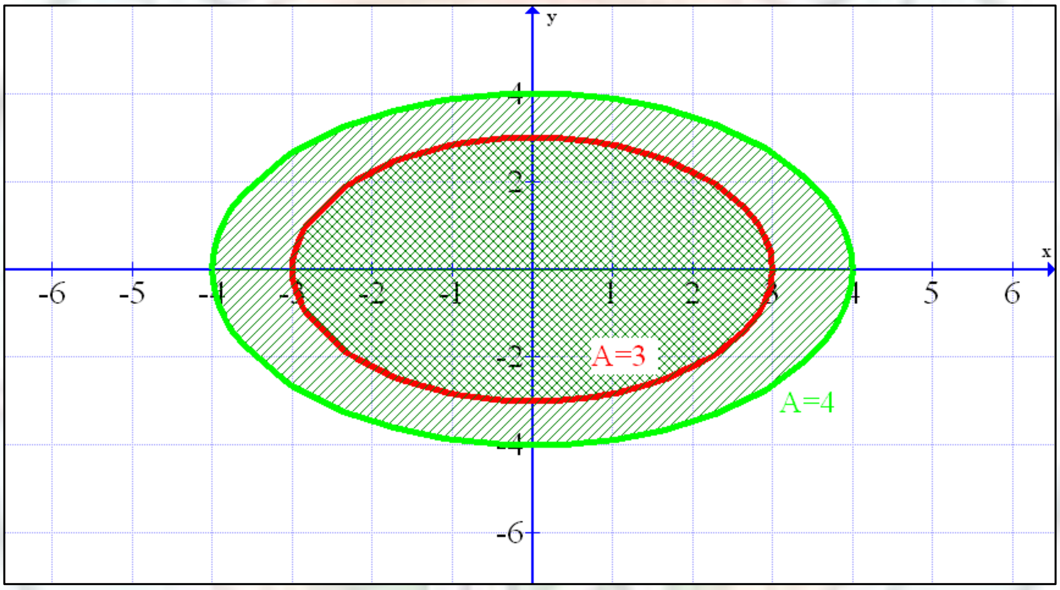
$$\int \alpha^2 y dy = -\int x dx$$
$$\alpha^2 \frac{y^2}{2} = C - \frac{x^2}{2}$$
$$\alpha^2 y^2 + x^2 = A^2$$

where $A^2 = 2C$ is the arbitrary constant. This is an *implicit* solution for the DE.

We can look for the *explicit* solution of the form by solving for $y(x)$

$$y(x) = \pm\sqrt{\frac{A^2 - x^2}{\alpha^2}}$$

The general solution represents the family of ellipses as shown. Let $\alpha = 1$ then $y = \pm\sqrt{A^2 - x^2}$ and some representative curves would be



We can see that the interval of validity of the particular solution $y = \pm\sqrt{9 - x^2}$ for $A = 3$ is $x \in [-3,3]$ while that for the particular solution $y = \pm\sqrt{16 - x^2}$ for $A = 4$ is $x \in [-4,4]$.

Example 3.2.2 Solve the DE by separating variables

$$\boldsymbol{y' = \alpha^2 y}$$

and plot the family of curves obtained from the solution.

Solution: First we write it in the Differential Form

$$dy - \alpha^2 y dx = 0$$

By separating the variable we get

$$\frac{dy}{y} = \alpha^2 dx$$

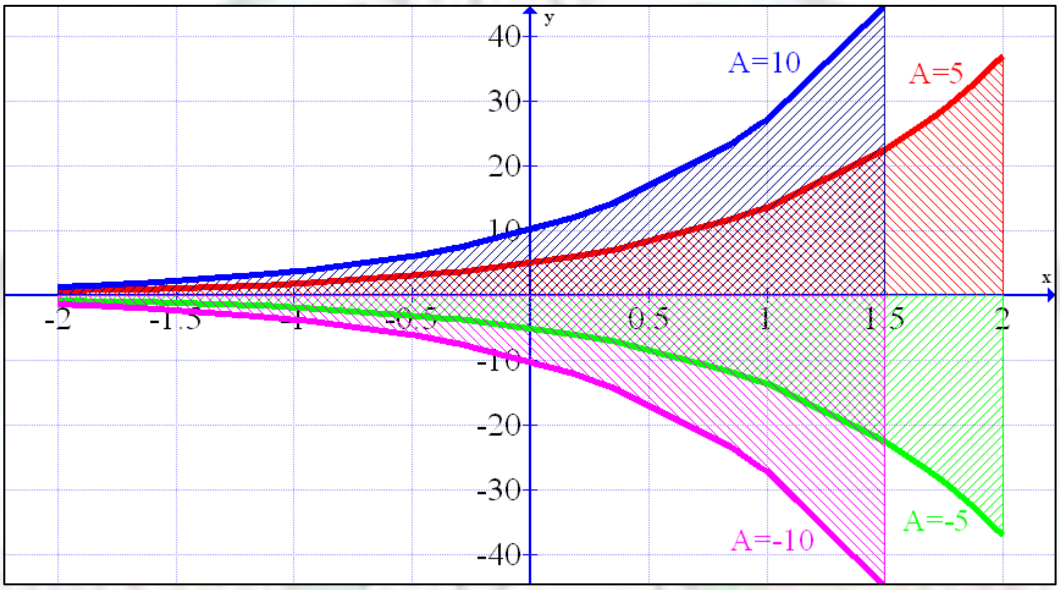
We can now integrate the equation

$$\int y^{-1}dy = \alpha^2 \int dx$$
$$\ln y = \alpha^2 x + C$$
$$y = e^{\alpha^2 x + C}$$

and write it in the explicit form

$$y(x) = Ae^{\alpha^2 x}$$

where $A = e^C$.
The general solution represents the family of curves as shown. Let $\alpha = 1$ then $y = Ae^x$ and some representative curves would be



We can see that the interval of validity of the particular solutions $y = \pm 5e^x$ for $A = \pm 5$ is $x \in (-\infty, \infty)$ and the same holds for the particular solutions $y = \pm 10e^x$ for $A = \pm 10$ is $x \in (-\infty, \infty)$.

Example 3.2.3 Solve the DE by separating variables

$$\boldsymbol{y' + \alpha^2 y^2 = 0}$$

Solution: We would like to note that the 1st order DE is not Linear but it is easily separable as

$$y' = -\alpha^2 y^2$$

which is in the *product* form. By separating the variable we get

$$\frac{dy}{y^2} = -\alpha^2 dx$$

and integrating we get

$$\int y^{-2}dy = -\alpha^2 \int dx$$

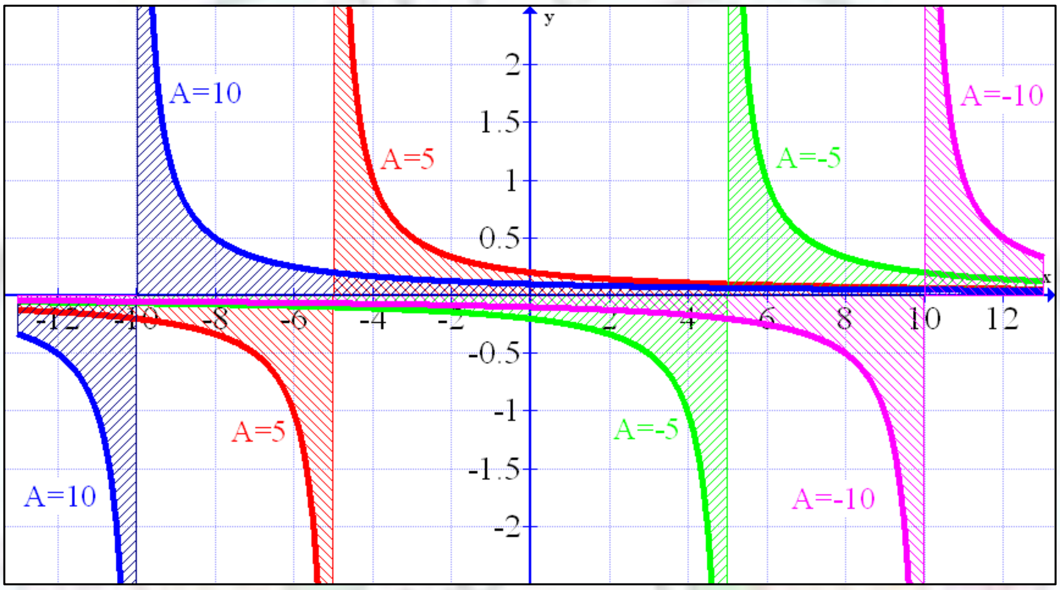
$$\frac{y^{-1}}{(-1)} = -\alpha^2 x + C$$

$$\alpha^2 x - \frac{1}{y} = C$$

which yields the *explicit* solution

$$y(x) = \frac{1}{(\alpha^2 x + A)}$$

where $A = -C$ is the arbitrary constant.
The general solution represents the family of ellipses as shown. Let $\alpha = 1$ then $y = \frac{1}{(x+A)}$ and some representative curves would be



We can see that the interval of validity of the particular solutions
$y = \frac{1}{(x-5)}$ for $A = -5$ is $x \in (-\infty, 5)$ & $x \in (5, \infty)$
$y = \frac{1}{(x+5)}$ for $A = 5$ is $x \in (-\infty, -5)$ & $x \in (-5, \infty)$
$y = \frac{1}{(x-10)}$ for $A = -10$ is $x \in (-\infty, 10)$ & $x \in (10, \infty)$ &
$y = \frac{1}{(x+10)}$ for $A = +10$ is $x \in (-\infty, -10)$ & $x \in (-10, \infty)$.

Example 3.2.4 Solve the DE by separating variables

$$\boldsymbol{y' - \alpha^2 y^2 = \beta^2}$$

Solution: We would like to note that the 1st order DE is also not Linear but it is separable as

$$y' = \beta^2 + \alpha^2 y^2$$

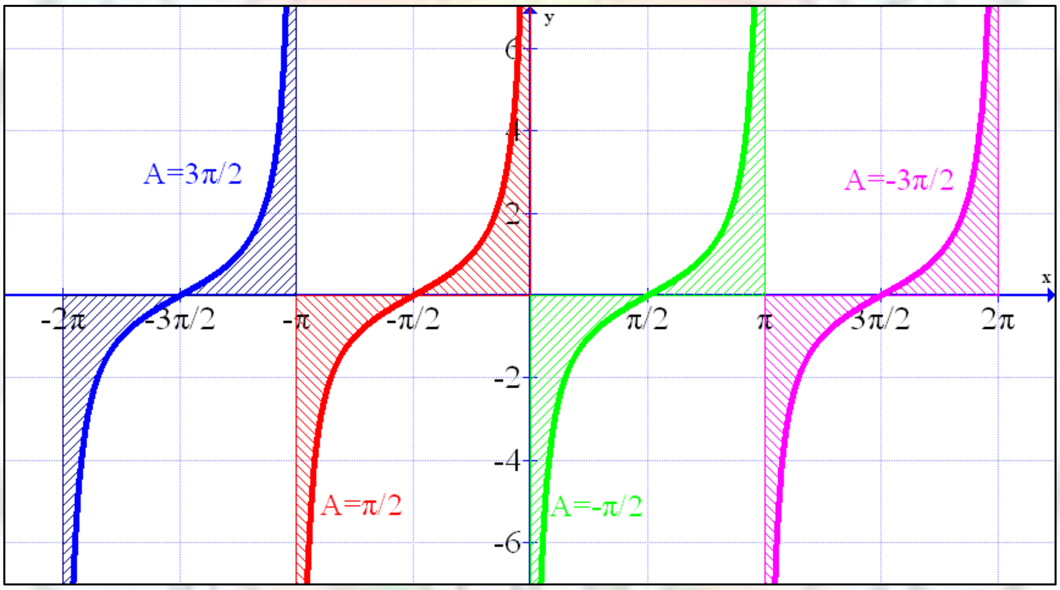
which is in the product form. By separating the variable we get

$$\frac{dy}{\beta^2 + \alpha^2 y^2} = dx$$
$$\frac{dy}{1 + (\alpha^2 y^2/\beta^2)} = \beta^2 dx$$
$$\int \frac{dy}{1 + (\alpha^2 y^2/\beta^2)} = \beta^2 \int dx$$
$$tan^{-1}\left(\frac{\alpha y}{\beta}\right) = \beta^2(x + C)$$
$$tan^{-1}\left(\frac{\alpha y}{\beta}\right) = \beta^2 x + A$$

which yields the *explicit* solution

$$y(x) = \frac{\beta}{\alpha}\tan(A + \beta^2 x)$$

where $A = \beta^2 C$ is the arbitrary constant.
The general solution represents the family of curves as shown. Let $\alpha = \beta = 1$ then $y = \tan(A + x)$ and some representative curves would be



We can see that the interval of validity of the particular solutions
$y = \tan\left(x + \frac{\pi}{2}\right)$ for $A = \frac{\pi}{2}$ is $x \in (-\pi, 0)$
$y = \tan\left(x - \frac{\pi}{2}\right)$ for $A = -\frac{\pi}{2}$ is $x \in (0, \pi)$
$y = \tan\left(x + \frac{3\pi}{2}\right)$ for $A = \frac{3\pi}{2}$ is $x \in (-2\pi, -\pi)$
$y = \tan\left(x - \frac{3\pi}{2}\right)$ for $A = -\frac{3\pi}{2}$ is $x \in (\pi, 2\pi)$.

Example 3.2.5 Solve the DE by separating variables

$$\boldsymbol{y - xy' = \alpha(1 - x^2 y')}$$

where $\alpha = 1$.

Solution: We would like to note that it is a 1st order Linear DE and can be written as

$$-xy' + x^2 y' = 1 - y$$

$$(-x + x^2)y' = (1 - y)$$
$$y' = (1 - y)(x^2 - x)^{-1}$$

which is in the product form. By separating the variable we get

$$\frac{dy}{(1-y)} = \frac{dx}{(x^2 - x)}$$
$$\frac{dy}{(y-1)} = -\frac{dx}{x(x-1)}$$
$$\frac{dy}{(y-1)} = \frac{(-1 + x - x)dx}{x(x-1)} = \left[\frac{1}{x} - \frac{1}{(x-1)}\right]dx$$
$$\int \frac{dy}{(y-1)} = \int \left[\frac{1}{x} - \frac{1}{(x-1)}\right]dx \to (A)$$

If $x > 1$ we can write the solution as

$$\ln|y-1| = \ln|x| - \ln|x-1| + C$$
$$\ln|y-1| = \ln\frac{x}{(x-1)} + C$$
$$\ln\left\{(y-1)\frac{(x-1)}{x}\right\} = C$$
$$(y-1)\frac{(x-1)}{x} = e^C$$

which yields the *explicit* solution

$$y(x) = \frac{Ax}{(x-1)} + 1$$

where $A = e^C$ is the arbitrary constant.

If $0 < x < 1$ we can write the solution (using the solution integral Eq. A) as

$$\ln|y-1| = \ln|x| + \ln|1-x| + C$$
$$\ln|y-1| = \ln x(1-x) + C$$
$$(y-1) = e^C x(1-x)$$

which yields the *explicit* solution

$$y(x) = Ax(1-x) + 1$$

where $A = e^C$ is the arbitrary constant.

If $x < 0$ we can write the solution (using the solution integral Eq. A) as

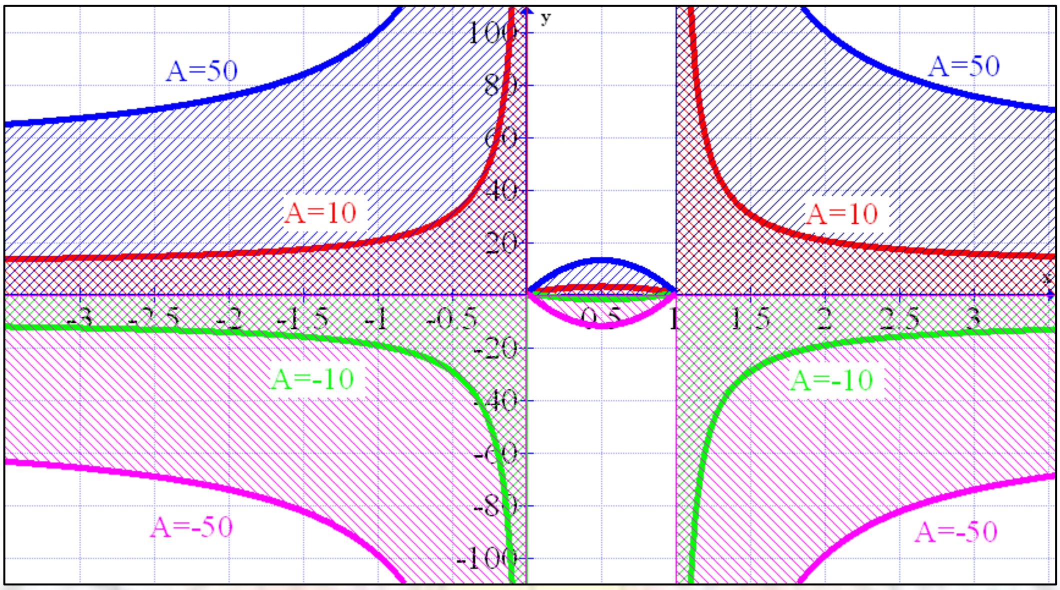
$$\ln|y-1| = -\ln|-x| + \ln|1-x| + C$$
$$\ln|y-1| = \ln\frac{(1-x)}{-x} + C$$
$$(y-1) = e^C\frac{(x-1)}{x}$$

which yields the *explicit* solution

$$y(x) = A\frac{(x-1)}{x} + 1$$

where $A = e^C$ is the arbitrary constant.

The general solution represents the family of curves as shown. Some representative curves would be



We can see that the interval of validity of the particular solutions

$y = \frac{10x}{(x-1)} + 1$ for $A = 10$ is $x \in (1, -\infty)$

$y = -\frac{10x}{(x-1)} + 1$ for $A = -10$ is $x \in (1, -\infty)$

$y = \frac{50x}{(x-1)} + 1$ for $A = 50$ is $x \in (1, -\infty)$

$y = -\frac{50x}{(x-1)} + 1$ for $A = -50$ is $x \in (1, -\infty)$.

and

$y = 10x(1-x) + 1$ for $A = 10$ is $x \in (0,1)$

$y = -10x(1-x) + 1$ for $A = -10$ is $x \in (0,1)$

$y = 50x(1-x) + 1$ for $A = 50$ is $x \in (0,1)$

$y = -50x(1-x) + 1$ for $A = -50$ is $x \in (0,1)$.

while

$y = 10\frac{(x-1)}{x} + 1$ for $A = 10$ is $x \in (-\infty, 0)$

$y = -10\frac{(x-1)}{x} + 1$ for $A = -10$ is $x \in (-\infty, 0)$

$y = 50\frac{(x-1)}{x} + 1$ for $A = 50$ is $x \in (-\infty, 0)$

$y = -50\frac{(x-1)}{x} + 1$ for $A = -50$ is $x \in (-\infty, 0)$.

Example 3.2.6 Solve the DE by separating variables

$$e^{\alpha x}dx - \beta y dy = 0$$

Solution: It is a 1st order Linear DE and can be written as

$$y' = \frac{e^{\alpha x}}{\beta y}$$
$$y' = e^{\alpha x}(\beta y)^{-1}$$

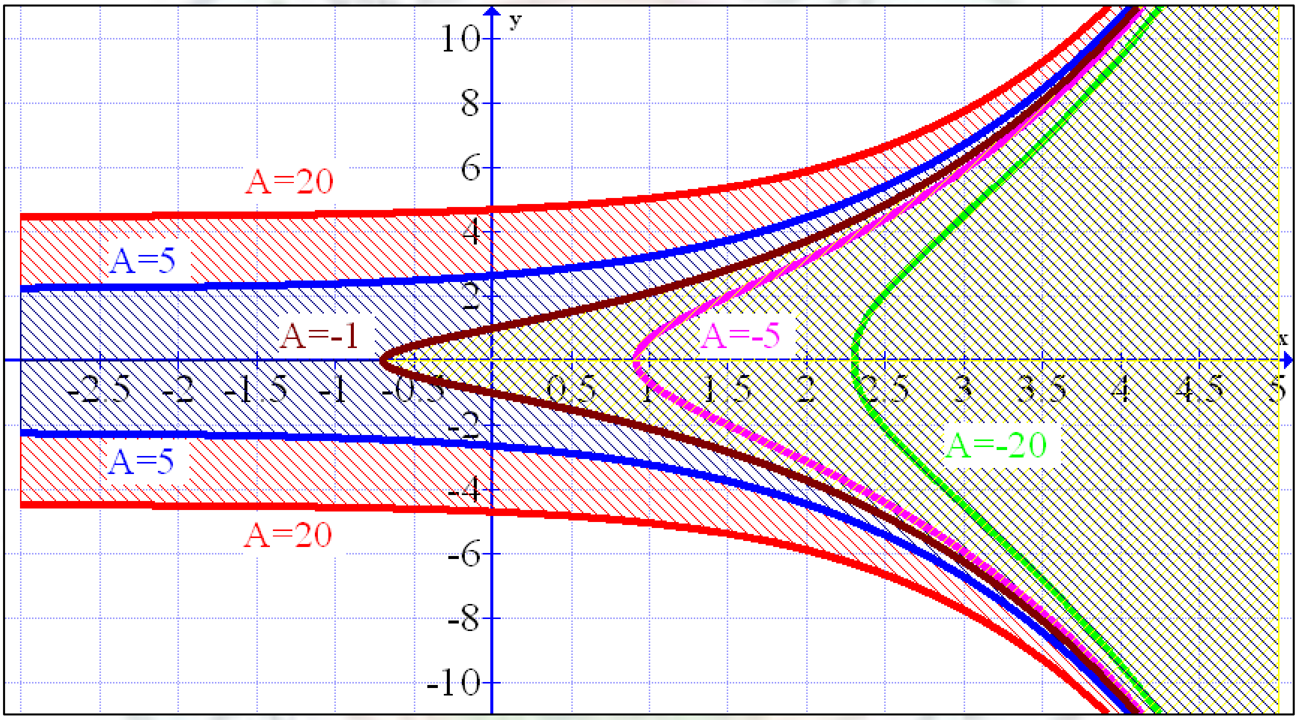
which is in the *product* form. By separating the variable we get

$$\beta y dy = e^{\alpha x} dx$$
$$\int \beta y dy = \int e^{\alpha x} dx$$
$$\beta \frac{y^2}{2} = \frac{e^{\alpha x}}{\alpha} + C$$

which yields the *explicit* solution

$$y(x) = \pm \sqrt{2 \frac{e^{\alpha x}}{\alpha\beta} + A}$$

where $A = 2C/\beta$ is the arbitrary constant.
The general solution represents the family of curves as shown. Let $\alpha = \beta = 1$ then $y = \pm\sqrt{2e^x + A}$ and some representative curves would be



We can see that the interval of validity of the particular solutions
$y = \pm\sqrt{2e^x + 20}$ for $A = 20$ is $x \in (-\infty, +\infty)$
$y = \pm\sqrt{2e^x + 5}$ for $A = 5$ is $x \in (-\infty, +\infty)$
$y = \pm\sqrt{2e^x - 1}$ for $A = -1$ is $x \in (-\ln 2, +\infty)$
$y = \pm\sqrt{2e^x - 5}$ for $A = -5$ is $x \in \left(\ln {}^5/_2, +\infty\right)$ &
$y = \pm\sqrt{2e^x - 20}$ for $A = -20$ is $x \in (\ln 10, +\infty)$.

3.3 Value Addition *(Let's try to look at some interesting real life DE and understand how we arrive at a solution)*

Example 3.3.1 Obtain the solution to the Banker's Equation where the amount

y, the interest rate r and the withdrawal rate w are determined by the DE

$$\frac{dy}{dt} = (r - w)y$$

Solution: By separating the variable we get

$$\frac{dy}{y} = (r - w)dt$$

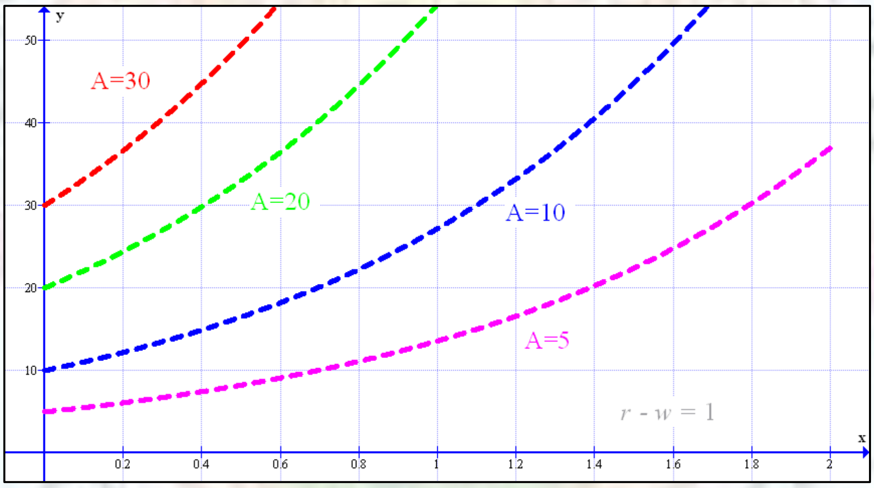
Integrating it we find

$$\int \frac{dy}{y} = \int (r - w)dt$$
$$\ln y = (r - w)t + C$$

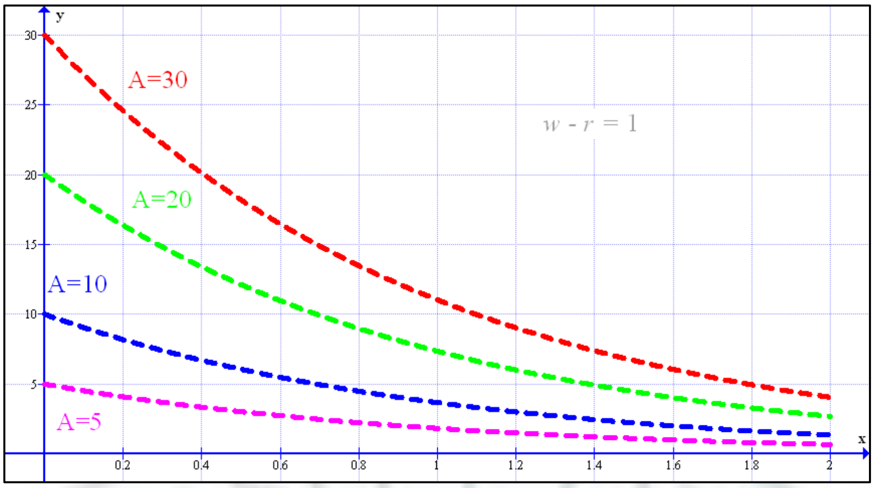
where $A = e^C$ is the arbitrary constant. The *explicit* solution for the DE is then

$$y(t) = Ae^{(r-w)t}$$

The general solution represents the family of exponentials as shown.
Let $r - w = 1$ then $y(t) = Ae^t$ and some representative curves would be



So we find that if the interest rate is greater than the withdrawal rate money would grow.
We can see that the interval of validity of the particular solution is $t > 0$.
Let $w - r = 1$ then $y(t) = Ae^{-t}$ and some representative curves would be



So we find that if the withdrawal rate is greater than the interest rate money would reduce.
We can see that the interval of validity of the particular solution is $t > 0$.

Example 3.3.2 Let a tank contain 1000 *litre* of water in which 40kg of salt is dissolved. *5 litre* of brine, each containing *1kg* of dissolved salt runs into the tank per second. The mixture in the tank is kept homogeneous. The mixture now runs out of the tank at the same rate of *5 litre* per second. Find the amount of salt y at any time t in the tank.

Solution:
The salt in the tank changes at a rate = Salt inflow rate − Salt outflow rate

Salt inflow rate = 5 kg /sec as *(5 litre of brine, each containing 1kg of dissolved salt runs into the tank per second.)*

Salt outflow rate
Let $y(t)$ be the total amount of salt present at time t in the tank which always contains 1000 litre of water. Each litre in the tank contains $y(t)/1000$ kg salt. Thus 5 litre contains $y(t) * 5/1000$ kg salt. Therefore, the rate of change of salt in the tank is

$$\frac{dy}{y} = 5 - 0.0005\, y(t)$$
$$= -0.0005(\, y(t) - 1000)$$
$$\frac{dy}{y - 1000} = -0.0005\, dt$$

Integrating it we find

$$\int \frac{dy}{y - 1000} = -0.0005 \int dt$$
$$\ln |y - 1000| = -0.0005\, t + C$$

where $A = e^C$ is the arbitrary constant. The *explicit* solution for the DE is then

$$y(t) - 1000 = Ae^{-0.0005\, t}$$

Using the initial condition at t=0 the amount is 40 kg we get

$$y(t=0) = 40$$

So

$$40 - 1000 = A$$

Hence

$$y(t) = 1000 + 960\, e^{-0.0005\, t}$$

Example 3.3.3 If the temperature 'T' of a system at $t = 0$ is $64℃$. After 2 hours ($t = 2$), temperature becomes $55℃$. Find the temperature of the system at $t = 8$ hours . Assume temperature of the surrounding to be $34℃$.

Solution:

The time rate of change of temperature is

$$T' = \frac{dT}{dt}$$

And

$$T' \alpha\, (T - Ts)$$

Ts being the temperature of the surrounding, then

$$T - Ts = \frac{dT}{dt}$$

$$\frac{dT}{dt} = k(T - 34)$$

or

$$\frac{dT}{T-34} = k\, dt$$

Integrating it we find

$$\int \frac{dT}{T-34} = k\, t + C$$

$$\ln|T - 34| = Ae^{kt}$$

where $A = e^C$ is the arbitrary constant. The *explicit* solution for the DE is then

$$T - 34 = Ae^{kt}$$

Using the initial condition at t=0 the temperature is $64℃$ we get

$$64 - 34 = A$$

Hence

$$T = 34 + 30\, e^{k\, t}$$

Using $T(t=2) = 55℃$ since after 2 hours the temperature is $55℃$ we get

$$55 = 34 + 30\, e^{k\, 2}$$
$$k = -0.1783$$

And so finally

$$T = 34 + 30\, e^{-0.1783\, t}$$

So the temperature after 8 hours is

$$T = 34 + 30\, e^{-0.1783*8} = 41.2\ ℃$$

Summary

Standard and Differential Form of the Differential Equation

The general first order differential equation is given by $F(x, y, y') = 0$ where x and y are independent and dependent variables respectively and y' is of ***degree 1***.

- We write the Standard Form as $y' = f(x, y)$
- It is to be noted that not all equations can be written in the standard form.
- The Differential Form is written as $M(x, y)dx + N(x, y)dy = 0$

Separable Equations

- A separable differential equation can be written in the form $y' = P(x)Q(y)$, as a product of two functions which are independent of each other.

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