Discipline Course-I
Semester -I
Paper: Mathematical PhysicsI IA
Lesson: Partial Differentiation
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Chapter 11: Partial Differentiation

Learning Objectives

The student will understand the meaning of

- Partial Derivatives of a function
- How to write the Total Differentials?
- Understand the Theorems on Total Differentials
    - nature of the function for the Existence of its Total Differential
    - constraint on the function for the Existence of its Total Differential Equation
- Differentiate between the Exact & In-exact Differentials for a function of two or more variables
    - the primary condition on the function for the differential to be exact
    - to know that the integral of *exact differential* are path independent
- Partial Differentials & Power Series

A function f written as a power series about point (h, k)

Partial Differentiation

11.1Partial Derivatives

We have seen that often derivatives determining rates have to be solved in physics. Additional feature of derivatives are that they can be used to determine extreme values of a function. However, more often a physical quantity may depend on two or more variables and therefore there is a need to define rates with partial dependence on one of the variables with all other variables treated as constants.

The ordinary derivatives of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant is called the partial derivative of the function with respect to the variable. So by this definition, if f is a function of two variables $x \,\&\, y$ then

$$\frac{\partial f}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} \equiv \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

provided such limit exists at the point (x, y).

To understand this graphically, let's suppose a physical quantity z depends on two variables $x \,\&\, y$

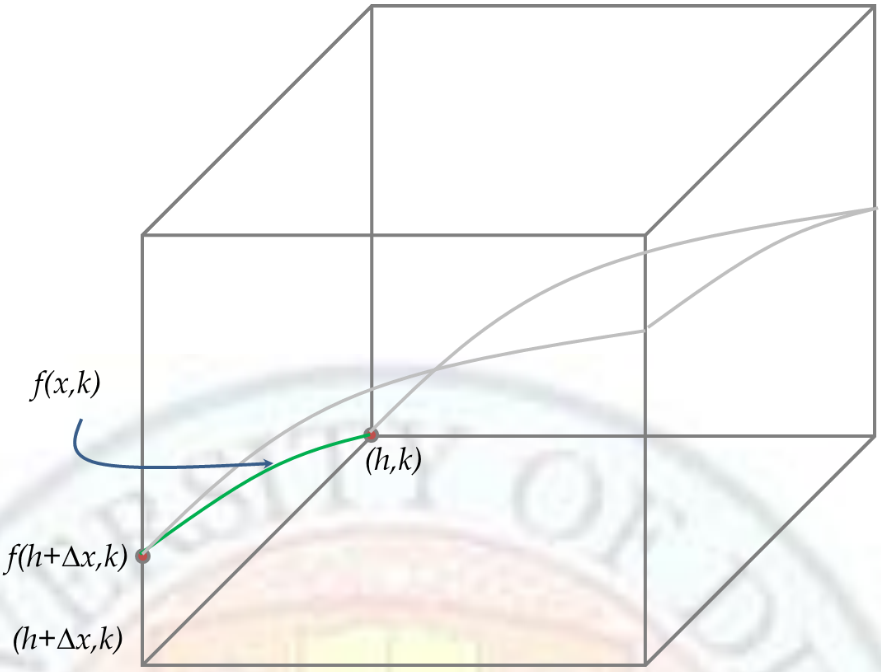
$$z = f(x, y)$$

Now suppose that we are at point (h, k):
Point 1 we can give the value to $y = k$ so that the *function* $z = f(x, k)$ will be curve in the $(x\,, z)$ plane. We can define the rate change z with x and write it as

$$\frac{\partial}{\partial x} z \equiv \frac{d}{dx}\, f(x, k)$$

and the change in z due to variation of x by an amount Δx would be

$$\Delta z = \frac{\partial z}{\partial x}\, \Delta x = \frac{d}{dx}\, f(x, k)\, \Delta x$$

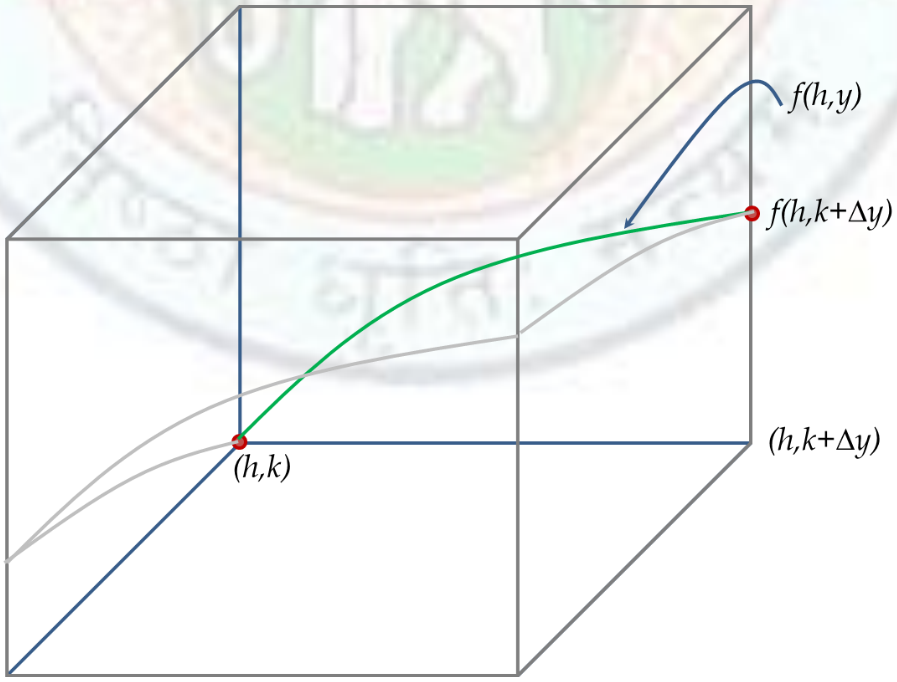


Point 2 we can give the value to $x = h$ so that the *function* $z = f(h, y)$ will be curve in the $(y\,, z)$ plane. We can define the rate change of z with y and write it as

$$\frac{\partial}{\partial y} z \equiv \frac{d}{dy}\, f(k, y)$$

and the change in z due to variation of y by an amount Δy would be

$$\Delta z = \frac{\partial z}{\partial y}\, \Delta y = \frac{d}{dy}\, f(h, y)\, \Delta y$$



Example 11.1.1 Find the partial derivatives of the function $f = x^a y^b$ at any point (x, y). What is its value if $a = 2$ & $b = 4$ and the point is $(-1, 2)$

Solution: The *partial derivative with respect to the variable* x will be

$$\frac{\partial f}{\partial x} = \frac{d}{dx} f\bigg|_{y=C} = \frac{d}{dx}\{x^a C^b\} = C^b \frac{d}{dx}\{x^a\} = C^b a x^{a-1}$$

Now putting back y for C we get

$$\frac{\partial f}{\partial x} = y^b a x^{a-1} = a x^{a-1} y^b$$

Similarly, the *partial derivative with respect to the variable* y will be

$$\frac{\partial f}{\partial y} = \frac{d}{dy} f\bigg|_{x=C} = \frac{d}{dx}\{C^a y^b\} = C^a \frac{d}{dx}\{y^b\} = C^a b y^{b-1}$$

Now putting back x for C we get

$$\frac{\partial f}{\partial y} = x^a b y^{b-1} = b x^a y^{b-1}$$

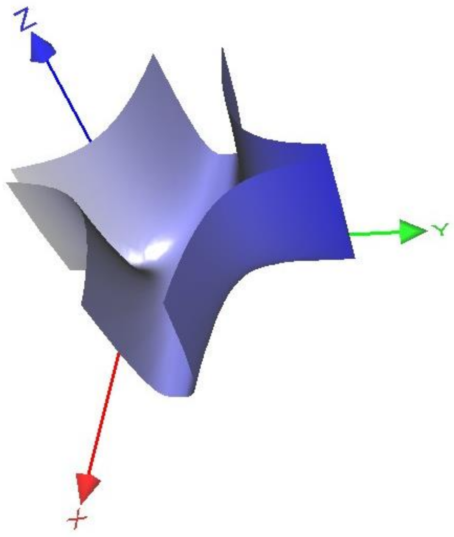
For the given values of $a = 2$ & $b = 4$, the *partial derivatives at point* (x, y) would be

$$\frac{\partial f}{\partial x} = 2xy^4 \;\&\; \frac{\partial f}{\partial y} = 4x^2y^3$$

So for the point $(-1,2)$

$$\frac{\partial f}{\partial x} = -32 \;\&\; \frac{\partial f}{\partial y} = 32$$

Let's see it on the graph:



Example 11.1.2 **Find the partial derivatives of the function $f = x^a + y^b$ at any point (x, y). What is its value if $a = 1$ & $b = 2$ and the point is** $(-1, 3)$

Solution: The *partial derivative with respect to the variable* x will be

$$\frac{\partial f}{\partial x} = \frac{d}{dx} f\Big|_{y=C} = \frac{d}{dx}\{x^a + C^b\} = ax^{a-1}$$

Similarly, the *partial derivative with respect to the variable* y will be

$$\frac{\partial f}{\partial y} = \frac{d}{dy} f\Big|_{x=C} = \frac{d}{dy}\{C^a + y^b\} = by^{b-1}$$

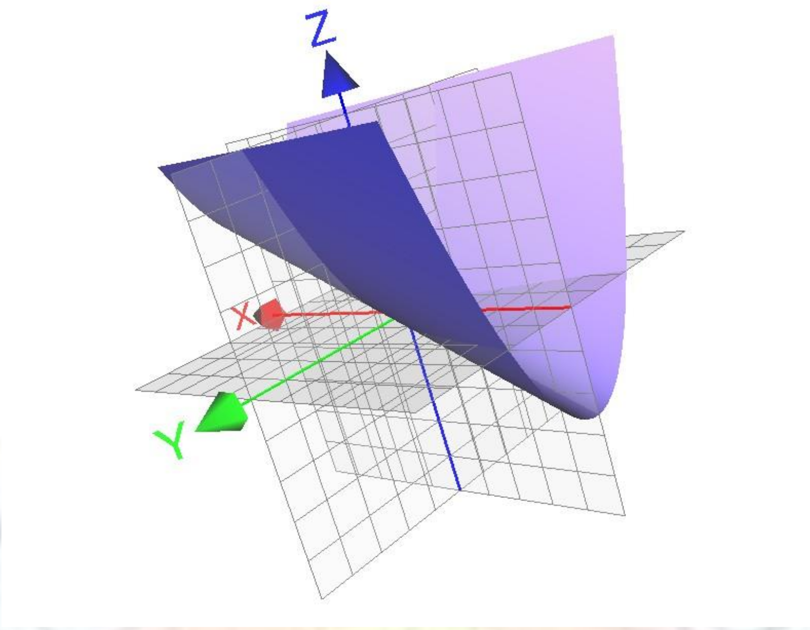
For the given values of $a = 1$ & $b = 2$, the *partial derivatives at point* (x, y) would be

$$\frac{\partial f}{\partial x} = 1 \;\&\; \frac{\partial f}{\partial y} = 2y$$

So for the point $(-1,2)$

$$\frac{\partial f}{\partial x} = 1 \;\&\; \frac{\partial f}{\partial y} = 4$$

Let’s see it on the graph:



Example 11.1.3 Find the partial derivatives of the function $f = \sqrt{x^a + y^b}$ at any point (x, y). What is its value if $a = 2$ & $b = 4$ and the point is $(-1, 3)$

Solution: The *partial derivative with respect to the variable* x will be

$$\frac{\partial f}{\partial x} = \frac{d}{dx} f\Big|_{y=C} = \frac{d}{dx}\left\{\sqrt{x^a + C^b}\right\} = \frac{1}{2\sqrt{x^a + C^b}} \frac{d}{dx}\{x^a + C^b\} == \frac{a x^{a-1}}{2\sqrt{x^a + C^b}}$$

Now putting back y for C we get

$$\frac{\partial f}{\partial x} = \frac{a x^{a-1}}{2\sqrt{x^a + y^b}}$$

Similarly, the *partial derivative with respect to the variable* y will be

$$\frac{\partial f}{\partial y} = \frac{d}{dy} f\Big|_{x=C} = \frac{d}{dy}\left\{\sqrt{C^a + y^b}\right\} = \frac{1}{2\sqrt{C^a + y^b}} \frac{d}{dx}\{C^a + y^b\} == \frac{b y^{b-1}}{2\sqrt{C^a + y^b}}$$

Now putting back x for C we get

$$\frac{\partial f}{\partial y} = \frac{b y^{b-1}}{2\sqrt{x^a + y^b}}$$

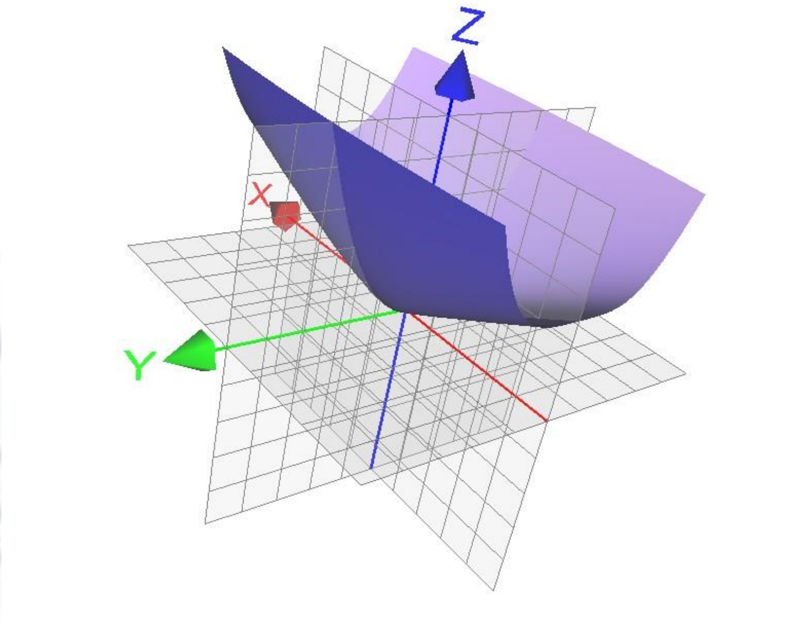
For the given values of $a = 2$ & $b = 4$, the *partial derivatives at point* (x, y) would be

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^4}} \;\&\; \frac{\partial f}{\partial y} = \frac{2y^3}{\sqrt{x^2 + y^4}}$$

So for the point $(-1,2)$

$$\frac{\partial f}{\partial x} = \frac{-1}{\sqrt{17}} \;\&\; \frac{\partial f}{\partial y} = \frac{16}{\sqrt{17}}$$

Let's see it on the graph:



The higher order partial derivative can be similarly defined with some of them as follows

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \;\&\; f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \;\&\; f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Example 11.1.4 Find the partial derivatives of the function $f = 2x^3y^2 + 3x^2y^3$ at any point (x, y). What is its value at the point $(-1, 3)$

Solution: The *partial derivatives* would be

$$\frac{\partial f}{\partial x} = 6x^2y^2 + 6xy^3 \,, \frac{\partial f}{\partial y} = 4x^3y + 9x^2y^2$$

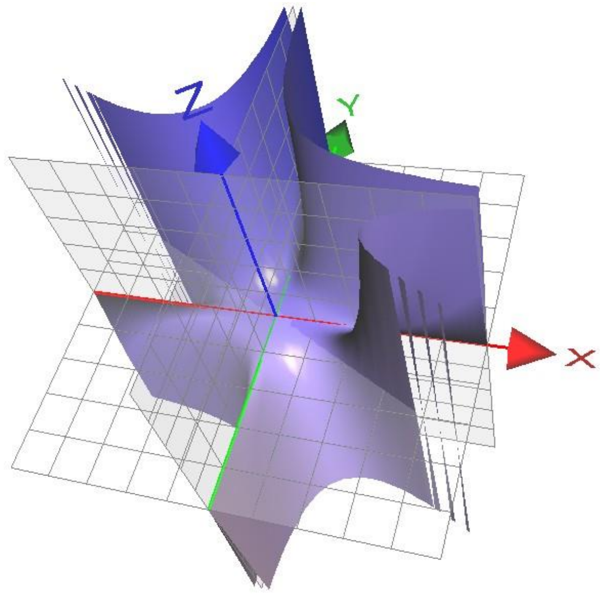
$$f_{xx} = \frac{\partial}{\partial x}(6x^2y^2 + 6xy^3) = 12xy^2 + 6y^3 \;\&\; f_{yy} = \frac{\partial}{\partial y}(4x^3y + 9x^2y^2) = 4x^3 + 18x^2y$$

$$f_{yx} = \frac{\partial}{\partial y}(6x^2y^2 + 6xy^3) = 12x^2y + 18xy^2 \;\&\; f_{xy} = \frac{\partial}{\partial x}(4x^3y + 9x^2y^2) = 12x^2y + 18xy^2$$

We should note here that $f_{yx} = f_{xy}$.

So for the point $(-1,2)$

$$\frac{\partial f}{\partial x} = -24\,, \frac{\partial f}{\partial y} = 28\,, f_{xx} = 0\,, f_{yy} = 32\ \&\ f_{yx} = -48 = \ f_{xy}$$



11.2Total Differentials

Let Δx and Δy be very small increments given to the variables x and y respectively. With these increments the value of x changes to $x + \Delta x$ and that of y changes to $y + \Delta y$. A function $f(x, y)$which depends on both the variables is than expected would take a new value $f(x + \Delta x, y + \Delta y)$, then the change in increment in function Δf

$$\Delta f = f(x + \Delta x, y + \Delta y) - f(x, y)$$

If the function f has *continuous first partial derivatives* then

$$\Delta f = \frac{\partial z}{\partial x}\,\Delta x + \epsilon_1 \Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_2 \Delta y$$

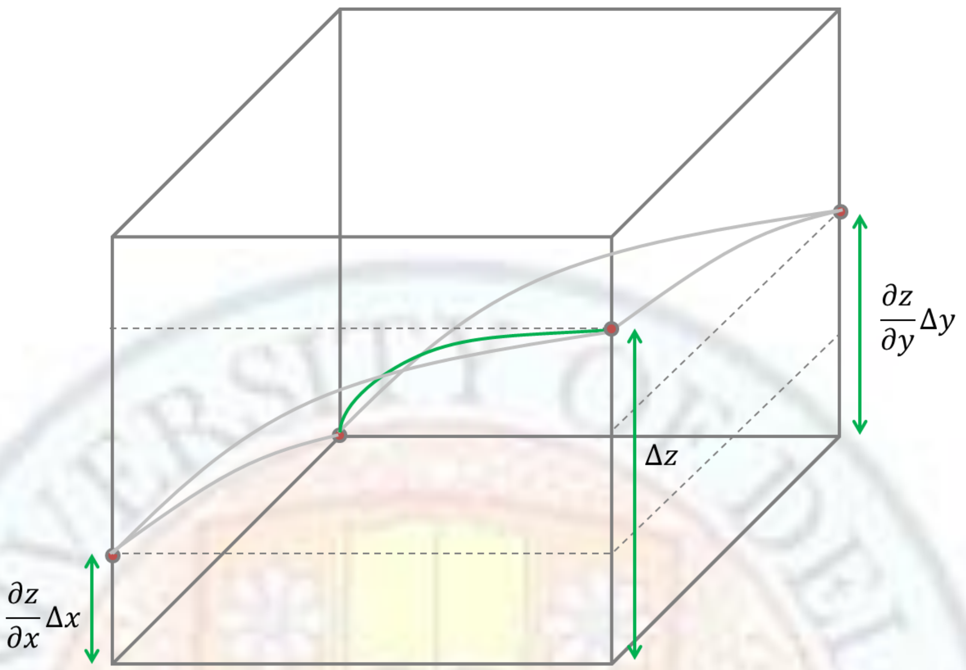
and under the limit $\Delta x \to 0\ \&\ \Delta y \to 0$ we get $\epsilon_1 \to 0\ \&\ \epsilon_2 \to 0$. Hence,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

This is known as the *Total Differential or Principal Part of the Differential or just the Differential of the function* f.

Graphically, Point 1 & Point 2 above tells us that the change in z due to variations of $x\ \&\ y$ together would be

$$\Delta z = \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y}\,\Delta y = \frac{d}{dx}\,f(x,k)\Delta x + \frac{d}{dy}\,f(h,y)\,\Delta y$$



To conclude, we say that the function f is dependent only on two variables x &y; that $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are to be read as partial derivatives of the function with respect to x &y respectively and that the total differential / principal part of the differential is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

if $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are continuous.

11.3 Theorems

11.3.1 Theorem on Existence: Total Differential

Theorem 1

If f is a function of n (independent or dependent) variables x_i and have continuous partial derivatives in some region R then

$$df = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}dx_i = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n$$

11.3.2 Theorem on Existence: Total Differential Equation

Theorem 2

If f is a function of n (independent or dependent) variables x_i having continuous partial derivatives in some region R and $f = C$, a constant then

$$df = 0$$
$$\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n = 0$$

and so all variables cannot be independent.

Example 11.3.1 Find the *differential of the function* $f = Ax^a e^{bxy}$.

Solution: The *partial derivatives* would be

$$\frac{\partial f}{\partial x} = Aax^{a-1} \times e^{bxy} + Ax^a \times e^{bxy}by = A\left(\frac{a}{x} + by\right)x^a e^{bxy}$$
$$\frac{\partial f}{\partial y} = Ax^a \times e^{bxy}bx = Abx^{a+1}e^{bxy}$$

And so the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = A\left(\frac{a}{x} + by\right)x^a e^{bxy}dx + Abx^{a+1}e^{bxy}dy$$

Note that

$$df = Ax^a e^{bxy}\left\{\left(\frac{a}{x} + by\right)dx + bxdy\right\} == Ax^a e^{bxy}\left\{\frac{a}{x}dx + b(ydx + xdy)\right\}$$

Example 11.3.2 Show that the *differential* $(3x^2y - 2y^2)dx + (x^3 - 4xy + 6y^2)dy$ is can be written as *differential of the function* $f = x^3y - 2xy^2 + 2y^3 + A$ where A is a constant.

Solution: The differential $(3x^2y - 2y^2)dx + (x^3 - 4xy + 6y^2)dy$ can be rewritten as

$$3x^2ydx - 2y^2dx + x^3dy - 4xydy + 6y^2dy$$

By some rearrangements the differential becomes

$$(3x^2ydx + x^3dy) - (2y^2dx + 4xydy) + 6y^2dy = (ydx^3 + x^3dy) - (2y^2dx + 2xdy^2) + 2dy^3$$

$$= d(x^3y) - 2d(xy^2) + 2d(y^3) = d(x^3y - 2xy^2 + 2y^3)$$

Thus, a function $x^3y - 2xy^2 + 2y^3$ has the above as its differential. In general, a constant A being added brings no change in the differential so the required function will be

$$f = x^3y - 2xy^2 + 2y^3 + A$$

Here, we would like to emphasise that it won't be always possible to find such a function f for an arbitrary *differential*.

11.4 Exact & In-exact Differentials

The *differential* $M(x,y)dx + N(x,y)dy$ is said to be *Exact Differential* if there exists a function two variables $\varphi(x,y)$ such that

$$d\varphi = M(x,y)dx + N(x,y)dy$$

We have already seen that the primary condition to be able to find such a function $\varphi(x,y)$ is to check if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If this is found to be true then we get

$$\varphi(x,y) = \int M(x,y)dx\Big|_{y=constant} + \int N(x,y)dy\Big|_{terms\ not\ containing\ x}$$

It important to note that integral of *exact differential* are path independent and therefore for any closed curve

$$\oint d\varphi(x,y) = 0 = \oint Mdx + Ndy$$

Similarly, the *differential* $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ is said to be *Exact Differential* if there exists a function of three variables $\varphi(x,y,z)$ such that

$$d\varphi = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

We find that the primary condition to be able to find such a function $\varphi(x,y,z)$ is to check if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \ \& \ \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

If this is found to be true then we get $\varphi(x,y,z)$ by integrating

$$\frac{\partial \varphi}{\partial x} = P, \frac{\partial \varphi}{\partial y} = Q \ \& \ \frac{\partial \varphi}{\partial z} = R$$

We again assert that the integral of *exact differential* are path independent and therefore for any closed curve

$$\oint d\varphi(x,y,z) = 0 = \oint Pdx + Qdy + Rdz$$

11.5 Some more Theorems

11.5.2 Theorem on Existence: Total Differential

Theorem 3

An expression of the form $P(x,y)dx + Q(x,y)dy$ is the "exact" differential of a function $f(x,y)$ if and only if

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

Proof :

Necessity

If $Pdx+Qdy=df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$ *then* $P=\frac{\partial f}{\partial x}$ & $Q=\frac{\partial f}{\partial y}$. *Now assuming the continuity of partial derivatives, we get*

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\frac{\partial f}{\partial y}$$

which means

$$\frac{\partial}{\partial y}P=\frac{\partial}{\partial x}Q$$

Sufficiency

If $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ *then by Green's Theorem the integral* $\oint_C(Pdx+Qdy)=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=0$ *and the integral* $\int(Pdx+Qdy)$ *is path independent. So we can define a function* $f(x,y)=\int(Pdx+Qdy)$ *and thus it's differential, which means*

$$df=Pdx+Qdy$$

11.5.3 Theorem on Existence: Total Differential

Theorem 4

An expression of the form $P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz$ *is the "exact" differential of a function* $f(x,y)$ *if and only if*

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}\ \&\ \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}$$

Proof :

Necessity

If $Pdx+Qdy+Rdz=df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$ *then* $P=\frac{\partial f}{\partial x},Q=\frac{\partial f}{\partial y}\ \&\ R=\frac{\partial f}{\partial z}$. *Now assuming the continuity of partial derivatives, we get*

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\frac{\partial f}{\partial y},\frac{\partial}{\partial z}\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\frac{\partial f}{\partial z}\ \&\ \frac{\partial}{\partial x}\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}\frac{\partial f}{\partial x}$$

which means

$$\frac{\partial}{\partial y}P=\frac{\partial}{\partial x}Q,\frac{\partial}{\partial z}Q=\frac{\partial}{\partial y}R\ \&\ \frac{\partial}{\partial x}R=\frac{\partial}{\partial z}P$$

Sufficiency

The integral $\oint_C(Pdx+Qdy+Rdz)$ *can be written as* $\oint_C\vec{A}\cdot d\vec{l}$ *where* $\vec{A}=P\hat{\imath}+Q\hat{\jmath}+R\hat{k}$ & $dl=dx\hat{\imath}+dy\hat{\jmath}+dz\hat{k}$. *Now, if* $\frac{\partial}{\partial y}P=\frac{\partial}{\partial x}Q,\frac{\partial}{\partial z}Q=\frac{\partial}{\partial y}R\ \&\ \frac{\partial}{\partial x}R=\frac{\partial}{\partial z}P$ *then* $\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=0,\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}=0\ \&\ \frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}=0$ *and so* $\nabla\times\vec{A}=0$. *By Stokes' Theorem the integral* $\oint_C\vec{A}\cdot d\vec{l}=\iint_R(\nabla\times\vec{A})\cdot d\vec{s}=0$ *and the integral* $\int(Pdx+Qdy+Rdz)$ *is path independent. So we can define a function* $f(x,y,z)=\int(Pdx+Qdy+Rdz)$ *and thus it's differential, which means*

$$df=Pdx+Qdy+Rdz$$

Example 11.5.1 Show that the *differential* $(y+z)dx+(z+x)dy+(x+y)dz$ is exact and find the *function* f of which the above is differential.

Solution: Let's first write the *differential in standard form* as $Pdx + Qdy + Rdz$ so that

$$P = y + z$$
$$Q = z + x$$
$$R = x + y$$

The condition of exactness require the $\frac{\partial}{\partial y}P = \frac{\partial}{\partial x}Q$, $\frac{\partial}{\partial z}Q = \frac{\partial}{\partial y}R$ & $\frac{\partial}{\partial x}R = \frac{\partial}{\partial z}P$ so we check for the validity of these equations

$$\frac{\partial}{\partial y}P = \frac{\partial}{\partial x}Q \quad \rightarrow 1 = 1$$
$$\frac{\partial}{\partial z}Q = \frac{\partial}{\partial y}R \quad \rightarrow 1 = 1$$
$$\frac{\partial}{\partial x}R = \frac{\partial}{\partial z}P \quad \rightarrow 1 = 1$$

The *differential* $(y + z)dx + (z + x)dy + (x + y)dz$ is therefore *Exact*.
Now by some rearrangements the differential becomes

$$(ydx + xdy) + (zdy + ydz) + (xdz + zdx)$$
$$= d(xy) + d(yz) + d(zx)$$
$$= d(xy + yz + zx)$$

Thus, a function $xy + yz + zx$ has the above as its differential. In general, a constant A being added brings no change in the differential so the required function will be

$$f = xy + yz + zx + A$$

Example 11.5.2 Show that the *differential* $z(1 - z^2)dx + zdy - (x + y + xz^2)dz$ is exact if it is divided by z^2 and find the *function* f of which the above is differential.

Solution: Let's first write the *differential in standard form* as $Pdx + Qdy + Rdz$ so that

$$P = z(1 - z^2)$$
$$Q = z$$
$$R = -(x + y + xz^2)$$

The condition of exactness require the $\frac{\partial}{\partial y}P = \frac{\partial}{\partial x}Q$, $\frac{\partial}{\partial z}Q = \frac{\partial}{\partial y}R$ & $\frac{\partial}{\partial x}R = \frac{\partial}{\partial z}P$ so we check for the validity of these equations

$$\frac{\partial}{\partial y}P = \frac{\partial}{\partial x}Q \quad \rightarrow 0 = 0$$
$$\frac{\partial}{\partial z}Q = \frac{\partial}{\partial y}R \quad \rightarrow 1 \neq -1$$
$$\frac{\partial}{\partial x}R = \frac{\partial}{\partial z}P \quad \rightarrow -1 = 1 - 3z^2$$

The *differential* in its present form is therefore *Not Exact*.

Now let's divide it by z^2, we thus get

$$\left(\frac{1}{z}-z\right)dx+\frac{1}{z}dy-\left(\frac{x}{z^2}+\frac{y}{z^2}+x\right)dz$$
$$P=\left(\frac{1}{z}-z\right)$$
$$Q=\frac{1}{z}$$
$$R=-\left(\frac{x}{z^2}+\frac{y}{z^2}+x\right)$$

The condition of exactness require the $\frac{\partial}{\partial y}P=\frac{\partial}{\partial x}Q$, $\frac{\partial}{\partial z}Q=\frac{\partial}{\partial y}R$ & $\frac{\partial}{\partial x}R=\frac{\partial}{\partial z}P$ so we check for the validity of these equations again

$$\frac{\partial}{\partial y}P=\frac{\partial}{\partial x}Q \quad \rightarrow 0=0$$
$$\frac{\partial}{\partial z}Q=\frac{\partial}{\partial y}R \quad \rightarrow -\frac{1}{z^2}=-\frac{1}{z^2}$$
$$\frac{\partial}{\partial x}R=\frac{\partial}{\partial z}P \quad \rightarrow -\left(\frac{1}{z^2}+1\right)=\left(-\frac{1}{z^2}-1\right)$$

The *differential* $\left(\frac{1}{z}-z\right)dx+\frac{1}{z}dy-\left(\frac{x}{z^2}+\frac{y}{z^2}+x\right)dz$ is therefore *Exact*.
Now by some rearrangements the differential becomes

$$\left(\frac{1}{z}dx-zdx\right)+\frac{1}{z}dy+\left(-\frac{x}{z^2}dz-\frac{y}{z^2}dz-xdz\right)$$
$$=\left(\frac{1}{z}dx-\frac{x}{z^2}dz\right)+\left(\frac{1}{z}dy-\frac{y}{z^2}dz\right)+(-zdx-xdz)$$
$$=d\left(\frac{x}{z}\right)+d\left(\frac{y}{z}\right)-d(xz)$$
$$=d\left(\frac{x}{z}+\frac{y}{z}-xz\right)$$

Thus, a function $\frac{x}{z}+\frac{y}{z}-xz$ has the $\frac{1}{z^2}\{z(1-z^2)dx+zdy-(x+y+xz^2)dz\}$ as its differential but not $z(1-z^2)dx+zdy-(x+y+xz^2)dz$.

11.6 Partial Differentials & Power Series

An important feature of partial derivative is that a function of two or many variables could be written as a power series about point (h,k) by using partial derivatives.The function f at (x,y) is then

$$f(x,y)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^n f(x,y)\Big|_{(h,k)}$$

The first few terms would be

$$f(x,y) = f(h,k) + \Delta x \left.\frac{\partial f}{\partial x}\right|_{(h,k)} + \Delta y \left.\frac{\partial f}{\partial y}\right|_{(h,k)} + \frac{1}{2}\left\{(\Delta x)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(h,k)} + (\Delta y)^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{(h,k)} + 2\Delta x \Delta y \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(h,k)}\right\} + \cdots$$

$$f(x,y) = f(h,k) + \left\{\left.\frac{\partial f}{\partial x}\right|_{(h,k)} (x-h) + \left.\frac{\partial f}{\partial y}\right|_{(h,k)} (y-k)\right\} + \frac{1}{2}\left\{\left.\frac{\partial^2 f}{\partial x^2}\right|_{(h,k)} (x-h)^2 + \left.\frac{\partial^2 f}{\partial y^2}\right|_{(h,k)} (y-k)^2 + 2\Delta x \Delta y \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(h,k)} (x-h)(y-k)\right\} + \cdots$$

In short form it is written as

$$f(x,y) = f(h,k) + f_x(h,k) \times (x-h) + f_y(h,k) \times (y-k) + \frac{1}{2} f_{xx}(h,k) \times (x-h)^2 + \frac{1}{2} f_{yy}(h,k) \times (y-k)^2 + f_{xy}(h,k) \times (x-h)(y-k) + \cdots$$

If $h = k = 0$

$$f(x,y) = f(0,0) + f_x(0,0)\, x + f_y(0,0)\, y + \frac{1}{2} f_{xx}(0,0)\, x^2 + \frac{1}{2} f_{yy}(0,0)\, y^2 + f_{xy}(0,0)\, xy + \cdots$$

Example 11.6.1 Find the power series expansion for the function $z = \cos(x+y)$ about the origin.

Solution: Since $h = k = 0$ we can use the series

$$f(x,y) = f(0,0) + f_x(0,0)\, x + f_y(0,0)\, y + \frac{1}{2} f_{xx}(0,0)\, x^2 + \frac{1}{2} f_{yy}(0,0)\, y^2 + f_{xy}(0,0)\, xy + \cdots$$

Now

$$f(0,0) = \cos(0+0) = 1$$
$$f_x(0,0) = -\sin(0+0) = 0 \;\; \& \;\; f_y(0,0) = -\sin(0+0) = 0$$
$$f_{xx}(0,0) = -\cos(0+0) = -1 \;\; \& \;\; f_{yy}(0,0) = -\cos(0+0) = -1$$
$$f_{xy}(0,0) = -\cos(0+0) = -1$$

So the first few terms of the series will be

$$f(x,y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2 - xy + \cdots$$

Example 11.6.2 Find the power series expansion for the function $z = e^{x-y}$ about the origin.

Solution: Since $h = k = 0$ we can use the series

$$f(x,y) = f(0,0) + f_x(0,0)\, x + f_y(0,0)\, y + \frac{1}{2} f_{xx}(0,0)\, x^2 + \frac{1}{2} f_{yy}(0,0)\, y^2 + f_{xy}(0,0)\, xy + \cdots$$

Now

$$f(0,0) = e^{0-0} = 1$$
$$f_x(0,0) = e^{0-0} = 1 \ \& \ f_y(0,0) = -e^{0-0} = -1$$
$$f_{xx}(0,0) = e^{0-0} = 1 \ \& \ f_{yy}(0,0) = e^{0-0} = 1$$
$$f_{xy}(0,0) = -e^{0-0} = -1$$

So the first few terms of the series will be

$$f(x,y) = 1 + x - y + \frac{1}{2}x^2 - \frac{1}{2}y^2 - xy + \cdots$$

Summary

Partial Derivatives

- If f is a function of two variables $x \,\&\, y$ then $\frac{\partial f}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$ & $\frac{\partial f}{\partial y} \equiv \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$ provided such limit exists at the point (x,y).

Total Differentials

- If the function f is dependent only on two variables $x \,\&\, y$ and if $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are continuous then the 'total differential' / 'principal part of the differential' is $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.

Theorems

- Theorem on Existence Total Differential
  If f is a function of n (independent or dependent) variables x_i and have continuous partial derivatives in some region R then
  $$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}dx_i = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n$$
- Theorem on Existence Total Differential Equation
  If f is a function of n (independent or dependent) variables x_i having continuous partial derivatives in some region R and $f = C$, a constant then
  $$df = 0$$
  $$\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n = 0$$
  and so all variables cannot be independent.

Exact & In-exact Differentials

- The *differential* $M(x,y)dx + N(x,y)dy$ is said to be *Exact Differential* if there exists a function two variables $\varphi(x,y)$ such that $d\varphi = M(x,y)dx + N(x,y)dy$
- The primary condition to be able to find such a function $\varphi(x,y)$ is
  $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
- The integral of *exact differential* are path independent and therefore for any closed curve

$$\oint d\varphi(x,y) = 0 = \oint Mdx + Ndy$$

- Similarly, the *differential* $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ is said to be *Exact Differential* if there exists a function of three variables $\varphi(x,y,z)$ such that

$$d\varphi = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

- We find that the primary condition to be able to find such a function $\varphi(x,y,z)$ is to check if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$,$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ & $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$
- Again the integral of *exact differential* are path independent and therefore for any closed curve

$$\oint d\varphi(x,y,z) = 0 = \oint Pdx + Qdy + Rdz$$

Partial Differentials & Power Series

- A function f of two or many variables could be written as a power series about point (h,k) by using partial derivatives

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^n f(x,y)\bigg|_{(h,k)}$$

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