Discipline Course-I
Semester -I
Paper: Mathematical PhysicsI IA
Lesson: Partial Differentiation & Maxima-Minima

Lesson Developer: Savinder Kaur
College/Department: SGTB Khalsa College, University of Delhi

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Chapter 12: Partial Differentiation & Maxima-Minima

Learning Objectives

In many cases of physics and otherwise we are interested to know the extreme values of the function; in this chapter the student learns about the behaviour of a function and in particular

- ***What is meant by Maxima & Minima of a Function?***
- ***How we calculate the Maxima & Minima of a Function and their importance?***
- ***How to calculate the Maxima & Minima of a Function under Constraints by***
    - ***Method of Elimination of a Variable***
    - ***Method of Lagrange Multiplier***

Partial Differentiation & Maxima-Minima

14.1 Maxima & Minima of a Function

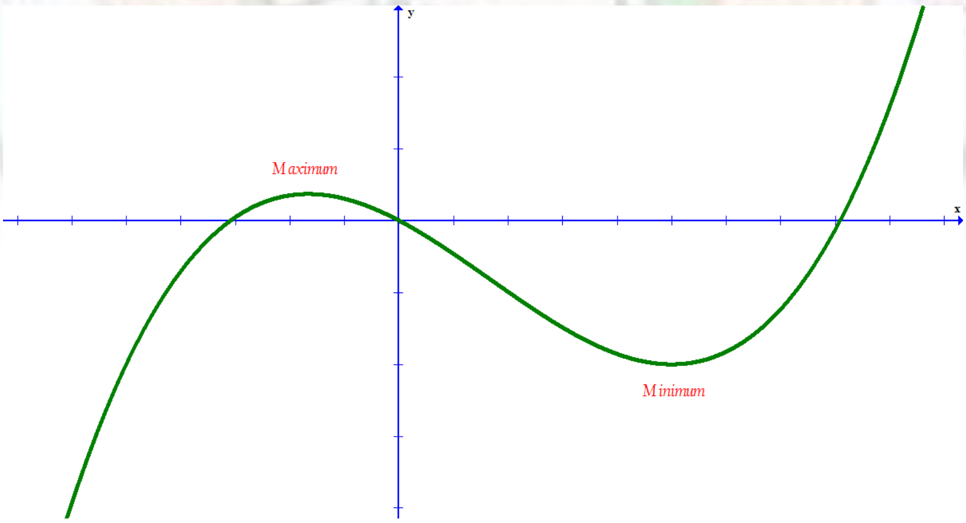
You may remember that a function $f(x)$ will represent a curve $y = f(x)$ in the (x, y) coordinate plane. It may have any arbitrary shape but there may exist a point(s) (x_m, y_m) which may be the lowest point (minimum) or the highest point (maximum) of the curve. These points could be identified by the well-known condition that at these point(s)

$$\left.\frac{d}{dx}f\right|_{x=x_m} = 0$$

To further pinpoint whether the point (x_m, y_m) is a maximum or minimum we need to check the sign of the second derivative $\left.\frac{d^2}{dx^2}f\right|_{x=x_m}$ at these points. We recall the rules

$$\left.\frac{d^2}{dx^2}f\right|_{x=x_m} > 0 \quad \rightarrow \quad Minimum$$

$$\left.\frac{d^2}{dx^2}f\right|_{x=x_m} < 0 \quad \rightarrow \quad Maximum$$



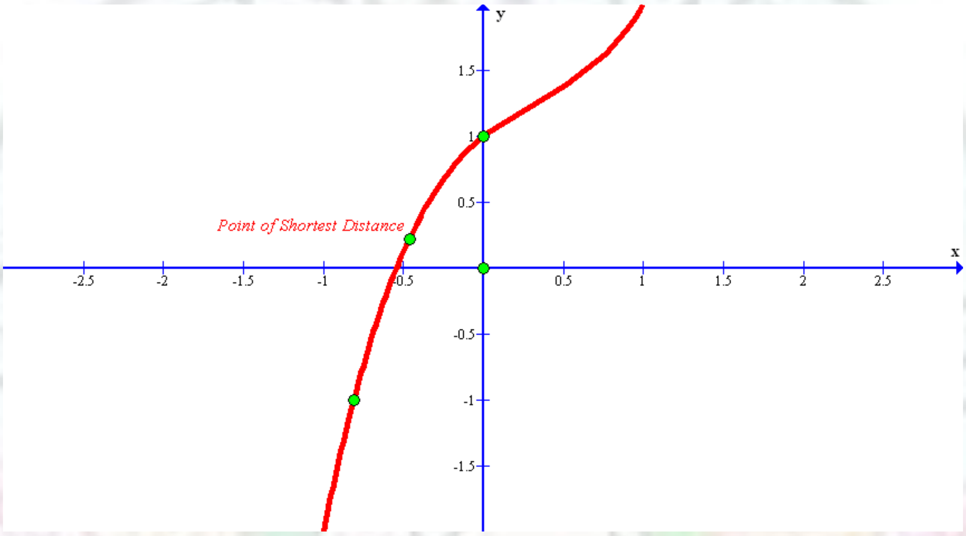
14.2 Maxima & Minima of a Function under Constraints

Situations may arise where we need to find the extreme values of the function under various constraints. Here, we wish to learn techniques to tackle such problems.

14.2.1 Method of Elimination of a Variable

Suppose your GPS shows that you are stranded at some place having the coordinates (x_0, y_0). The GPS map shows a road meandering around you fitting a form of a curve $y = g(x)$. The distance between you and any point (x, y) along the road would then be

$$f = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$



If you wish to find the shortest distance to the road then your problem would be to minimize "f" which is a function of (x, y). Clearly, the y values are constrained by the values of x in the form of the equation of the road-curve $y = g(x)$. Therefore, we can replace y in the expression for the distance f to get

$$f = \sqrt{[x - x_0]^2 + [g(x) - y_0]^2}$$

This extreme value of f can then be found by putting the condition

$$\frac{d}{dx} f \bigg|_{x=x_m} = 0$$

To make it simpler we note that

$$\frac{d}{dx} f^2 = 2f \frac{d}{dx} f = 0$$

So we get

$$2[x - x_0] + 2[g(x) - y_0]g'(x) = 0$$
$$[x - x_0] + [g(x) - y_0]g'(x) = 0$$

which yields some point on the road (x_m, y_m).
To find whether it is the shortest distance we have to check for

$$\left.\frac{d^2}{dx^2}f\right|_{x=x_m} > 0$$

To make it simpler we note that

$$\frac{d}{dx}\left(\frac{d}{dx}f^2\right) = \frac{d}{dx}\left(2f\frac{d}{dx}f\right) = 2\left(\frac{d}{dx}f\right)^2 + 2f\frac{d^2}{dx^2}f$$

$$\frac{d^2}{dx^2}f = \frac{1}{2f}\frac{d}{dx}\left(\frac{d}{dx}f^2\right)$$

So we get

$$\frac{d^2}{dx^2}f = \frac{1}{2f}\frac{d}{dx}\{2[x-x_0] + 2[g(x)-y_0]g'(x)\}$$

$$\frac{d^2}{dx^2}f = \frac{1}{2f}\{2 + 2g'(x) + 2[g(x)-y_0]g''(x)\}$$

$$\frac{d^2}{dx^2}f = \frac{\{1 + g'(x) + [g(x)-y_0]g''(x)\}}{f}$$

Since the distance f is assumed to be positive, the criterion reduces to

$$1 + g'(x)|_{x_m} + \left[g(x)|_{x_m} - y_0\right]g''(x)|_{x_m} > 0$$

at $x = x_m$. If this is indeed the case, then the point on the road (x_m, y_m) from now on will be called the *point of shortest distance*.

Example 12.2.1.1 Find the shortest distance from the origin to the curve $\boldsymbol{y = 1 + x}$.

Solution:

Step 1 The *distance function* f from the origin to be minimized is

$$f = \sqrt{x^2 + y^2}$$

The *constraint function* g is

$$y = g(x) = 1 + x$$

The *distance function* f can be rewritten as

$$f = \sqrt{x^2 + (1+x)^2}$$

Step 2 The extreme value of f would require

$$\frac{d}{dx}f = 0$$

$$\frac{1}{2\sqrt{x^2+(1+x)^2}}\frac{d}{dx}\{x^2+(1+x)^2\}=0$$

$$\frac{2x+2(1+x)}{2\sqrt{x^2+(1+x)^2}}=0$$

$$2x+1=0$$

$$x=-\frac{1}{2}$$

which implies the *point of shortest distance* is $\left(-\frac{1}{2},\frac{1}{2}\right)$.

Step 3 The minimum value of f would require

$$\frac{d^2}{dx^2}f=\frac{d}{dx}\frac{2x+1}{\sqrt{x^2+(1+x)^2}}$$

$$\frac{d^2}{dx^2}f=\frac{2}{\sqrt{x^2+(1+x)^2}}+(2x+1)\left(-\frac{1}{2}\right)\frac{2x+2(1+x)}{[x^2+(1+x)^2]^{\frac{3}{2}}}$$

$$\frac{d^2}{dx^2}f=\frac{2}{\sqrt{x^2+(1+x)^2}}-(2x+1)\frac{2x+1}{[x^2+(1+x)^2]^{\frac{3}{2}}}$$

$$\frac{d^2}{dx^2}f=\frac{2}{[x^2+(1+x)^2]^{\frac{3}{2}}}\{2[x^2+(1+x)^2]-(2x+1)^2\}$$

$$\frac{d^2}{dx^2}f=\frac{2}{[x^2+(1+x)^2]^{\frac{3}{2}}}\{2[2x^2+1+2x]-(4x^2+4x+1)\}$$

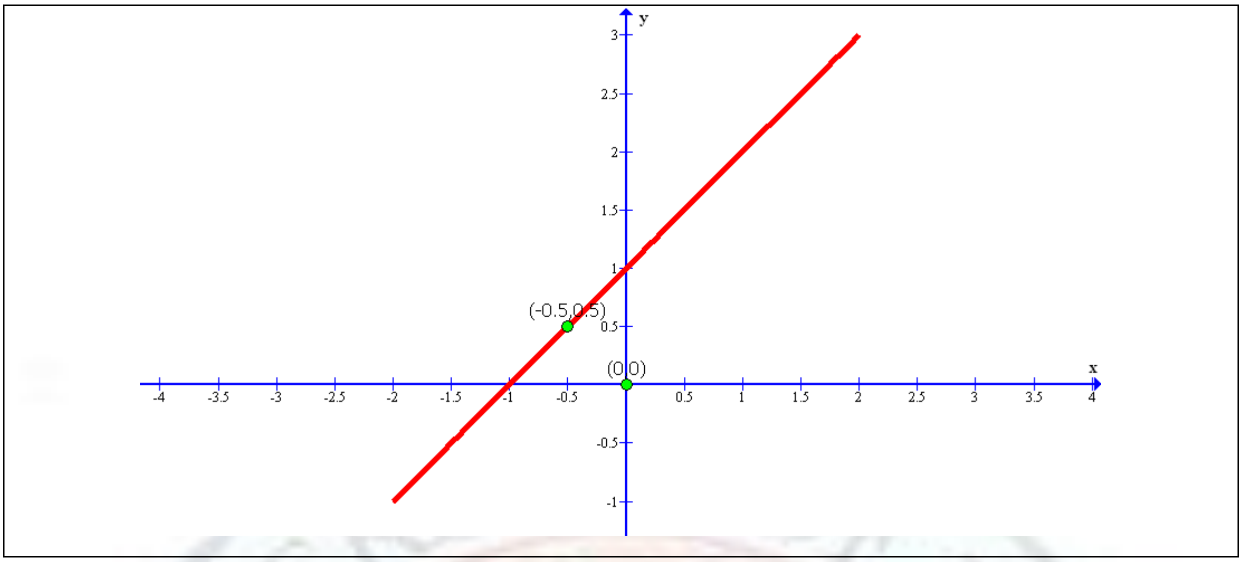
$$\frac{d^2}{dx^2}f=\frac{2}{[x^2+(1+x)^2]^{\frac{3}{2}}}$$

which means

$$\left.\frac{d^2}{dx^2}f\right|_{x=-1/2}>0$$

Thus, $\left(-\frac{1}{2},\frac{1}{2}\right)$ is indeed the *point of shortest distance*.

Let's see it on the graph:



Example 12.2.1.2 Find the shortest distance from the origin to the curve $y = 1 + x^2$.

Solution:

Step 1 The *distance function* g from the origin to be minimized is

$$f = \sqrt{x^2 + y^2}$$

The *constraint function* g is

$$y = g(x) = 1 + x^2$$

The *distance function* f can be rewritten as

$$f = \sqrt{x^2 + (1 + x^2)^2}$$

$$f = \sqrt{1 + 3x^2 + x^4}$$

Step 2 The extreme value of f would require $\frac{d}{dx} f = 0$ or equivalently

$$\frac{d}{dx} f^2 = 0$$

$$\frac{d}{dx}\{1 + 3x^2 + x^4\} = 0$$

$$6x + 4x^3 = 0$$

$$(3 + 2x^2)x = 0$$

$$x = 0, \sqrt{-\frac{3}{2}}$$

which implies the *point of shortest distance* is $(0,1)$.

Step 3 The minimum value of f would require

$$1 + g'(x) + [g(x) - y_0]g''(x) > 0$$

But

$$1 + g'(x) + [g(x) - y_0]g''(x) = 1 + 2x + [1 + x^2]x = 1 + 3x + x^3$$

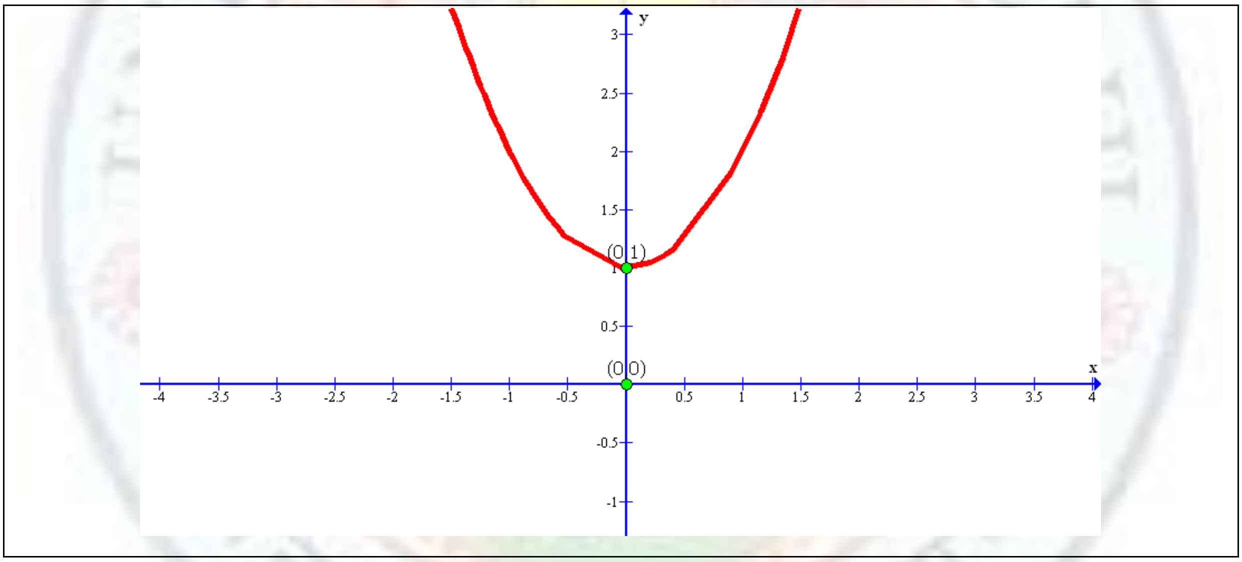
At the *point of shortest distance* $(0,1)$ this has a value

$$1 + g'(x) + [g(x) - y_0]g''(x) = 1$$

which means

$$\left.\frac{d^2}{dx^2} f\right|_{x=0} > 0$$

Let's see it on the graph:



Example 12.2.1.3 Find any local maxima or minima of the function $z = ax^2 + by^2$ subject to the constraint $\sqrt{a}x + \sqrt{b}y - \sqrt{c} = 0$.

Solution:
Step 1 The '*surface' function* f to be minimized is

$$f(x, y) = ax^2 + by^2$$

The *constraint 'plane'* g is

$$g(x, y) = \sqrt{a}x + \sqrt{b}y - \sqrt{c} = 0$$

The '*surface' function* f can be rewritten as

$$f = a\left(\frac{[\sqrt{c} - \sqrt{b}y]}{\sqrt{a}}\right)^2 + by^2 = [\sqrt{c} - \sqrt{b}y]^2 + by^2 = c - 2\sqrt{bc}y + 2by^2$$

Step 2 The extreme value of f would require

$$\frac{d}{dy}f = 0$$

$$\frac{d}{dy}\{c - 2\sqrt{bc}y + 2by^2\} = 0$$

$$-2\sqrt{bc} + 4by = 0$$

$$y = \frac{1}{2}\sqrt{\frac{c}{b}}$$

For which the x value would be

$$x = \sqrt{\frac{c}{a}} - \sqrt{\frac{b}{a}} \times \frac{1}{2}\sqrt{\frac{c}{b}} = \frac{1}{2}\sqrt{\frac{c}{a}}$$

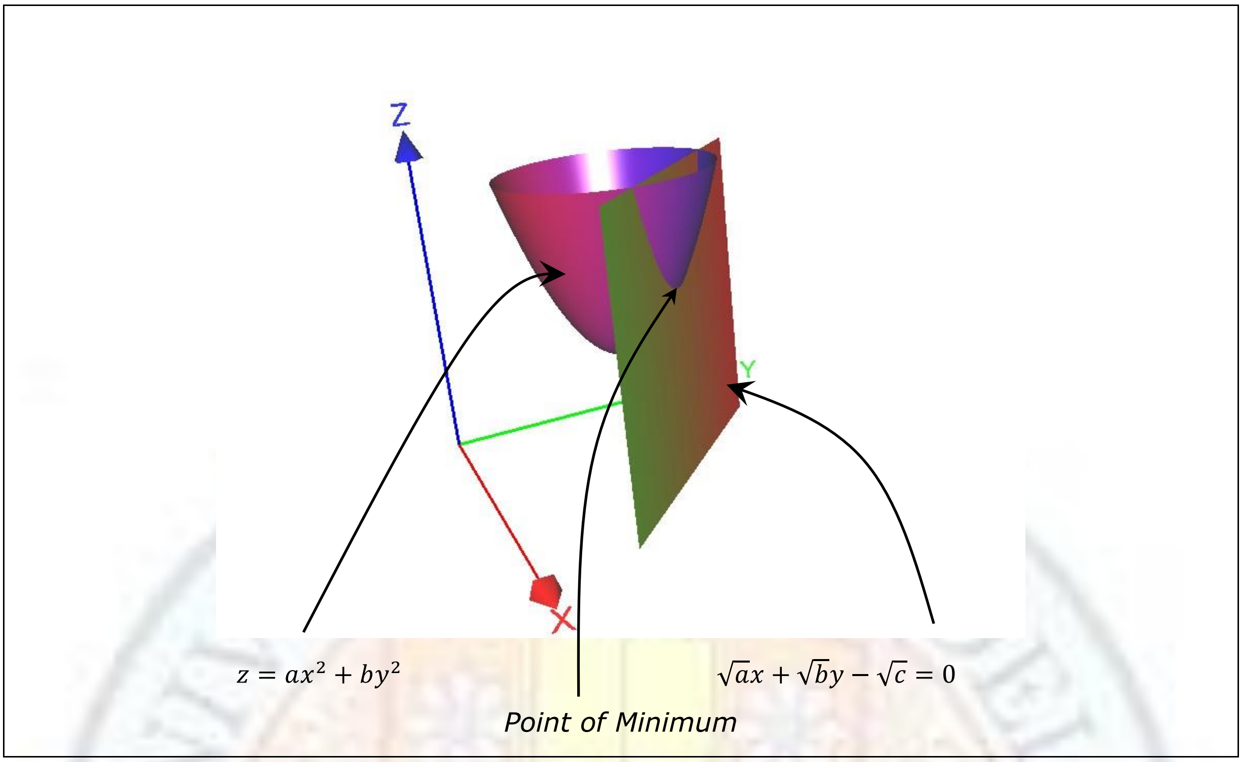
which implies the *point of extremum* is $\left(\frac{1}{2}\sqrt{\frac{c}{a}}, \frac{1}{2}\sqrt{\frac{c}{b}}\right)$.

Step 3 Lets find the second derivative of f

$$\frac{d^2}{dy^2}f = \frac{d}{dy}\left(\frac{d}{dy}f\right) = \frac{d}{dy}\left(-2\sqrt{bc} + 4by\right) = 4b$$

If $b > 0$ then $\frac{d^2}{dy^2}f > 0$, and so the point $\left(\frac{1}{2}\sqrt{\frac{c}{a}}, \frac{1}{2}\sqrt{\frac{c}{b}}\right)$ is the *point of minimum*.

Let's see it on the graph with $a = 1, b = 1 \,\&\, c = 1$:



Example 12.2.1.4 Find any local maxima or minima of the function $f = ax^2 + by^2 + cz^2$ subject to the constraint $\sqrt{a}x + \sqrt{b}y + \sqrt{c}z - \sqrt{d} = 0$.

Solution:

Step 1 The '*hypersurface' function* f to be minimized is

$$f(x, y, z) = ax^2 + by^2 + cz^2$$

The *constraint 'hyperplane'* g is

$$g(x, y, z) = \sqrt{a}x + \sqrt{b}y + \sqrt{c}z - \sqrt{d} = 0$$

The '*hypersurface' function* f can be rewritten as

$$f = a\left(\frac{\left[\sqrt{d} - \sqrt{b}y - \sqrt{c}z\right]}{\sqrt{a}}\right)^2 + by^2 + cz^2 = \left[\sqrt{d} - \sqrt{b}y - \sqrt{c}z\right]^2 + by^2 + cz^2$$

$$f = d + \left[\sqrt{b}y + \sqrt{c}z\right]^2 - 2\sqrt{d}\left[\sqrt{b}y + \sqrt{c}z\right] + by^2 + cz^2$$

$$f = d + 2by^2 + 2cz^2 + 2\sqrt{b}\sqrt{c}yz - 2\sqrt{d}\sqrt{b}y - 2\sqrt{d}\sqrt{c}z$$

Step 2 The extreme value of f would require

$$\frac{\partial}{\partial y}f = 0 \;\&\; \frac{\partial}{\partial z}f = 0$$

$$4by + 2\sqrt{b}\sqrt{c}z - 2\sqrt{d}\sqrt{b} = 0 \;\&\; 4cz + 2\sqrt{b}\sqrt{c}y - 2\sqrt{d}\sqrt{c} = 0$$

$$2\sqrt{b}y + \sqrt{c}z = \sqrt{d} \;\&\; 2\sqrt{c}z + \sqrt{b}y = \sqrt{d}$$

$$3\sqrt{c}z = \sqrt{d} \;\&\; 3\sqrt{b}y = \sqrt{d}$$

$$z = \frac{1}{3}\sqrt{\frac{d}{c}} \;\&\; y = \frac{1}{3}\sqrt{\frac{d}{b}}$$

For which the x value would be

$$x = \frac{\left[\sqrt{d} - \sqrt{b}y - \sqrt{c}z\right]}{\sqrt{a}} = \frac{\left[\sqrt{d} - \frac{1}{3}\sqrt{d} - \frac{1}{3}\sqrt{d}\right]}{\sqrt{a}} = \frac{1}{3}\sqrt{\frac{d}{a}}$$

which implies the *point of extremum* is $\left(\frac{1}{3}\sqrt{\frac{d}{a}}, \frac{1}{3}\sqrt{\frac{d}{b}}, \frac{1}{3}\sqrt{\frac{d}{c}}\right)$.

Step 3 Lets find the second derivatives of f

$$\frac{\partial^2}{\partial y^2} f = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f\right), \frac{\partial^2}{\partial z^2} f = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} f\right) \;\&\; \frac{\partial^2}{\partial yz} f = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} f\right)$$

$$\frac{\partial^2}{\partial y^2} f = \frac{\partial}{\partial y}\left(4by + 2\sqrt{b}\sqrt{c}z - 2\sqrt{d}\sqrt{b}\right), \frac{\partial^2}{\partial z^2} f = \frac{\partial}{\partial z}\left(4cz + 2\sqrt{b}\sqrt{c}y - 2\sqrt{d}\sqrt{c}\right) \;\&\; \frac{\partial^2}{\partial y \partial z} f = \frac{\partial}{\partial y}\left(4cz + 2\sqrt{b}\sqrt{c}y - 2\sqrt{d}\sqrt{c}\right)$$

$$\frac{\partial^2}{\partial y^2} f = 4b, \frac{\partial^2}{\partial z^2} f = 4c \;\&\; \frac{\partial^2}{\partial y \partial z} f = 2\sqrt{b}\sqrt{c}$$

We then find that

$$\left(\frac{\partial^2}{\partial y^2} f\right)\left(\frac{\partial^2}{\partial z^2} f\right) - \left(\frac{\partial^2}{\partial y \partial z} f\right)^2 = 4b \times 4c - \left(2\sqrt{b}\sqrt{c}\right)^2 = 4b \times 4c - 4bc = 12bc$$

which means that if $b, c > 0 \; or \; b, c < 0$ then

$$\left(\frac{\partial^2}{\partial y^2} f\right)\left(\frac{\partial^2}{\partial z^2} f\right) - \left(\frac{\partial^2}{\partial y \partial z} f\right)^2 > 0$$

and so *point of extremum* is $\left(\frac{1}{3}\sqrt{\frac{d}{a}}, \frac{1}{3}\sqrt{\frac{d}{b}}, \frac{1}{3}\sqrt{\frac{d}{c}}\right)$ is the *point of minimum*.

14.2.2 Method of Lagrange Multiplier

It may happen that the variable y cannot be reduced to a function of x from the constraint equation in an explicit form $y = g(x)$. Therefore, a general method is required to solve such problems.

Let $f(x, y)$ be the *main function* for which we want to determine the *Local Extremes*

$$f = f(x, y)$$

Let us also assume that there is a *constraint* put onto the function $g(x,y)$ by another *constraint function* of the same variables

$$g = g(x,y) = 0$$

a) Since the *constraint function* implicitly makes the y variable dependent on the x variable, the *extremum* of *main function* g can then be found by putting the condition

$$\frac{d}{dx}f = 0$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

for some point set (x_m, y_m).

b) The *constraint function* $g = 0$ itself gives

$$\frac{d}{dx}g = 0$$

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx} = 0$$

for any point set (x,y). This will therefore be true for the point set (x_m, y_m).

The two relations must be true simultaneously, so we get

$$-\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = \left.\frac{dy}{dx}\right|_{x_m} = -\frac{\left(\frac{\partial g}{\partial x}\right)}{\left(\frac{\partial g}{\partial y}\right)}$$

$$\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right) - \left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) = 0$$

$$\{f, g\}_{x,y} = 0$$

A much more convenient way is to define the derivative as a parameter known as the *Lagrange Multiplier* λ. For this, the relation then can be rewritten as,

$$\lambda \equiv -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial g}{\partial x}\right)} = -\frac{\left(\frac{\partial f}{\partial y}\right)}{\left(\frac{\partial g}{\partial y}\right)}$$

Thus, if f is the *main function* and $g = 0$ is the *constraint function*, then the *point of extremum* (x_m, y_m) of the *main function* by imposing the two conditions

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}=0\ \&\ \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}=0$$

We now show that a *Lagrange Multiplier* λ can be so chosen such that the stationary values of the function

$$\varphi\equiv f+\lambda g$$

always satisfy the two conditions

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}=0\ \&\ \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}=0$$

To do this, let's define the function $\varphi\equiv f(x,y)+\lambda g(x,y)$ with both the variables $x\,\&\,y$ *being treated as independent.* Since $\varphi=\varphi(x,y)$ is dependent on both the variables $x\,\&\,y$ its extremums would require

$$\frac{\partial\varphi}{\partial x}=0\ \&\ \frac{\partial\varphi}{\partial y}=0$$

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}=0\ \&\ \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}=0$$

This method is even more powerful as it can be extended to two or more constraint functions. Hence, if f is the *main function* and $g_1=0\,\&\,g_2=0$ are two *constraint functions*, then the *point of extremum* (x_m,y_m) of the *main function* can be found by defining a function

$$\varphi\equiv f(x,y)+\lambda_1 g_1(x,y)+\lambda_2 g_2(x,y)$$

and demanding

$$\frac{\partial f}{\partial x}+\lambda_1\frac{\partial g_1}{\partial x}+\lambda_2\frac{\partial g_2}{\partial x}=0\ \&\ \frac{\partial f}{\partial y}+\lambda_1\frac{\partial g_1}{\partial y}+\lambda_2\frac{\partial g_2}{\partial y}=0$$

Example 12.2.2.1 Find the shortest distance from the origin to the curve $y=1+x$ using the *Lagrange Multiplier*.

Solution:

Step 1 The *distance function* f from the origin to be minimized is

$$f=\sqrt{x^2+y^2}$$

The *constraint function* g is

$$g(x) = y - x - 1 = 0$$

Step 2 The *Lagrange Multiplier* λ is so chosen that the function

$$\varphi = f + \lambda g = \sqrt{x^2 + y^2} + \lambda(y - x - 1)$$

satisfies

$$\frac{\partial}{\partial x}\varphi = 0\,, \frac{\partial}{\partial y}\varphi = 0\ \&\ \frac{\partial}{\partial \lambda}\varphi = 0$$

$$\left\{\frac{x}{\sqrt{x^2+y^2}} - \lambda\right\} = 0\,, \left\{\frac{y}{\sqrt{x^2+y^2}} + \lambda\right\} = 0\ \&\ y - x - 1 = 0$$

Since the 3rd relation is already true, we use the first two

$$x = \lambda\sqrt{x^2+y^2}\ \&\ y = -\lambda\sqrt{x^2+y^2}$$

$$y = -x$$

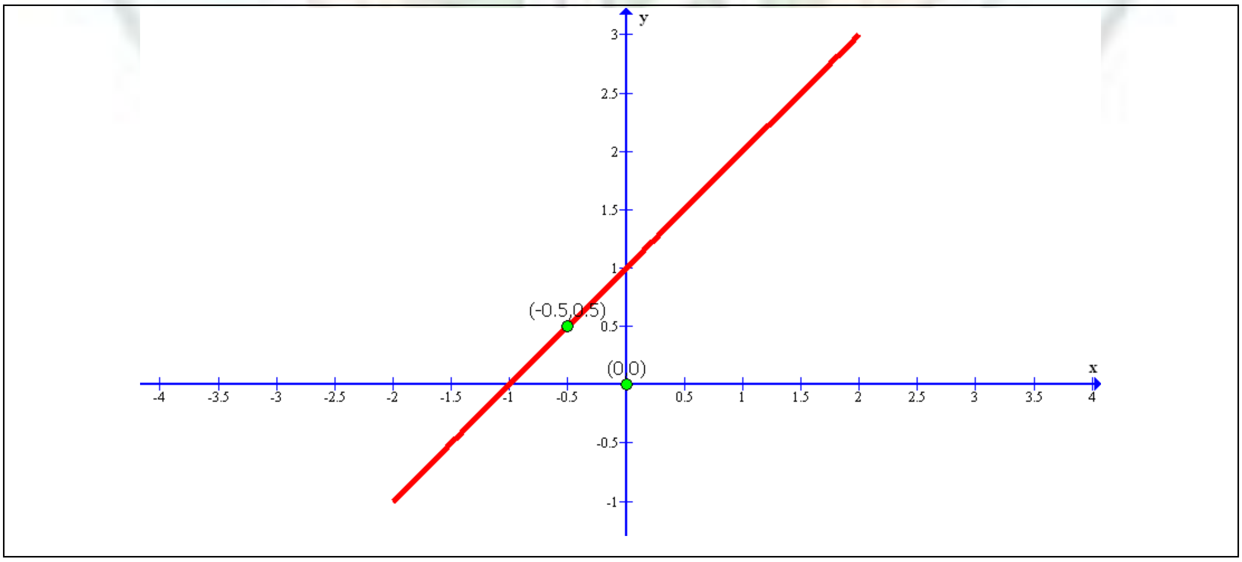
Step 3 Putting this in the *constraint function* g we get

$$(-x) - x - 1 = 0$$

$$x = -\frac{1}{2}\ \&\ x = +\frac{1}{2}$$

which means, as proved earlier, point $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is the *point of shortest distance* from the origin.

Let's see it on the graph:



Example 12.2.2.2 Find the shortest distance from the origin to the curve $y = 1 + x^2$ using the *Lagrange Multiplier*.

Solution:

Step 1 The *distance function* f from the origin to be minimized is

$$f = \sqrt{x^2 + y^2}$$

The *constraint function* g is

$$g(x) = y - x^2 - 1 = 0$$

Step 2 The *Lagrange Multiplier* λ is so chosen that the function

$$\varphi = f + \lambda g = \sqrt{x^2 + y^2} + \lambda(y - x^2 - 1)$$

satisfies

$$\frac{\partial}{\partial x}\varphi = 0\,, \frac{\partial}{\partial y}\varphi = 0 \;\&\; \frac{\partial}{\partial \lambda}\varphi = 0$$

$$\left\{\frac{x}{\sqrt{x^2 + y^2}} - 2\lambda x\right\} = 0\,, \left\{\frac{y}{\sqrt{x^2 + y^2}} + \lambda\right\} = 0 \;\&\; y - x^2 - 1 = 0$$

Since the 3rd relation is already true, we use the first two

$$x\left(1 - 2\lambda\sqrt{x^2 + y^2}\right) = 0 \;\&\; y = -\lambda\sqrt{x^2 + y^2}$$

$$x(1 + 2y) = 0$$

$$x = 0 \; or \; y = -\frac{1}{2}$$

However, as the curve doesn't pass through $y = -\frac{1}{2}$ the only proper choice is

$$x = 0$$

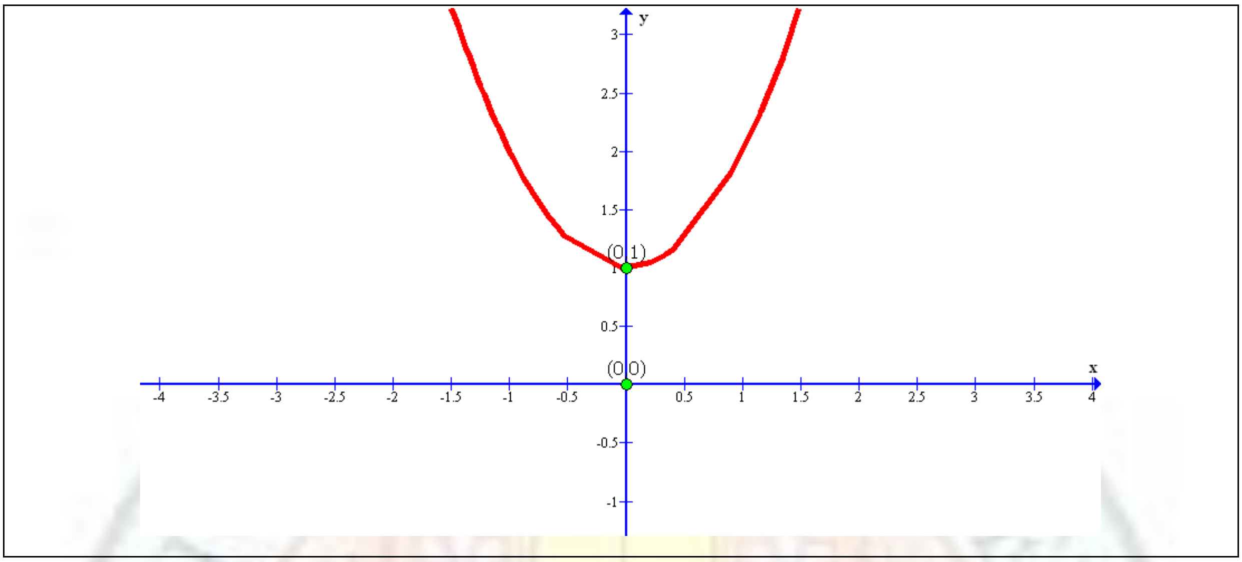
Step 3 Putting this in the *constraint function* g we get

$$y - (0)^2 - 1 = 0$$

$$y = 1$$

which means, as proved earlier, point $(0,1)$ is the *point of shortest distance* from the origin.

Let's see it on the graph:



Example 12.2.2.3 Find any local maxima or minima of the function $z = ax^2 + by^2$ subject to the constraint $\sqrt{a}x + \sqrt{b}y - \sqrt{c} = 0$ using the *Lagrange Multiplier*.

Solution:

Step 1 The '*surface' function* f to be minimized is

$$f(x, y) = ax^2 + by^2$$

The *constraint 'plane'* g is

$$g(x, y) = \sqrt{a}x + \sqrt{b}y - \sqrt{c} = 0$$

Step 2 The *Lagrange Multiplier* λ is so chosen that the function

$$\varphi = f + \lambda g = ax^2 + by^2 + \lambda\left(\sqrt{a}x + \sqrt{b}y - \sqrt{c}\right)$$

satisfies

$$\frac{\partial}{\partial x}\varphi = 0\,, \frac{\partial}{\partial y}\varphi = 0\ \&\ \frac{\partial}{\partial \lambda}\varphi = 0$$

$$2ax + \lambda\sqrt{a} = 0\,, 2by + \lambda\sqrt{b} = 0\ \&\ \sqrt{a}x + \sqrt{b}y - \sqrt{c} = 0$$

Since the 3rd relation is already true, we use the first two

$$2ax = -\lambda\sqrt{a}\ \&\ 2by = -\lambda\sqrt{b}$$

$$x = \sqrt{\frac{b}{a}}y$$

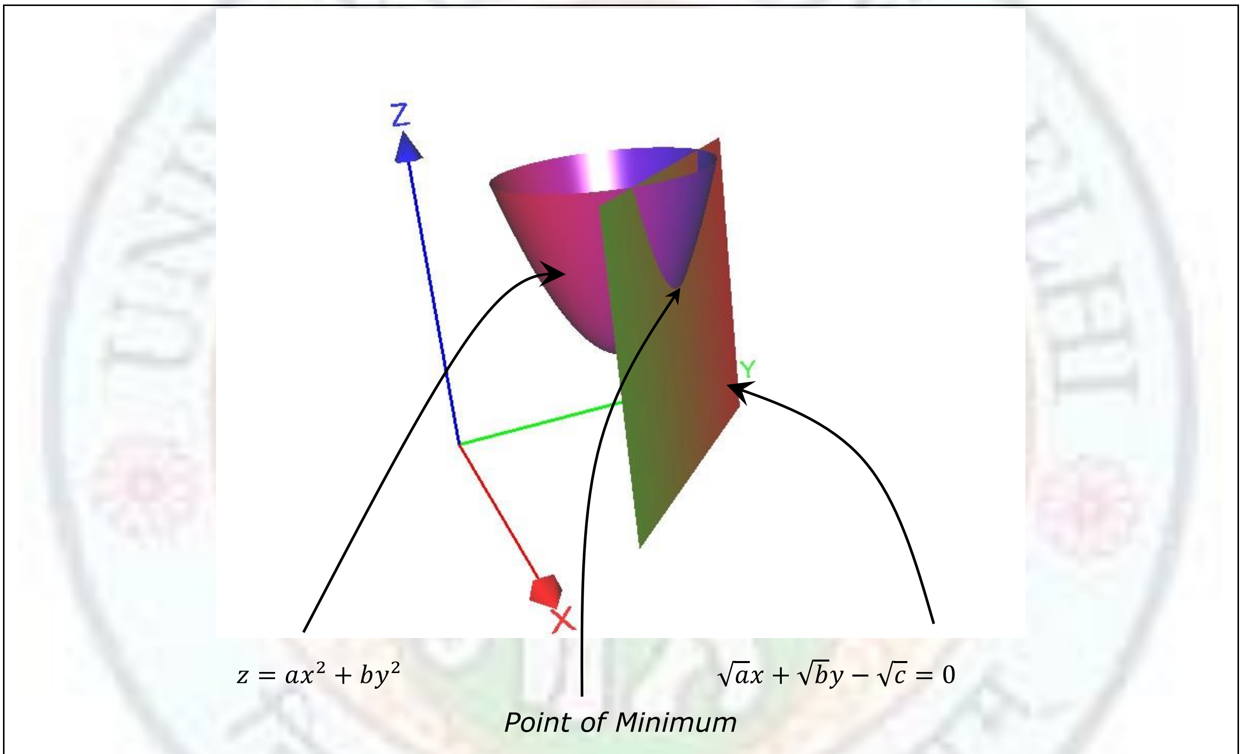
Step 3 Putting this in the *constraint function* g we get

$$\sqrt{a}\left(\sqrt{\frac{b}{a}}y\right) + \sqrt{b}y - \sqrt{c} = 0$$

$$y = \frac{1}{2}\sqrt{\frac{c}{b}} \;\&\; x = \frac{1}{2}\sqrt{\frac{c}{a}}$$

which means, as proved earlier, point $\left(\frac{1}{2}\sqrt{\frac{c}{a}}, \frac{1}{2}\sqrt{\frac{c}{b}}\right)$ is the *point of minimum*.

Let's see it on the graph with $= 1, b = 1 \;\&\; c = 1$:



Example 12.2.2.4 Find any local maxima or minima of the function $f = ax^2 + by^2 + cz^2$ subject to the constraint $\sqrt{a}x + \sqrt{b}y + \sqrt{c}z - \sqrt{d} = 0$ using the *Lagrange Multiplier*.

Solution:

Step 1 The '*hypersurface' function* f to be minimized is

$$f(x, y, z) = ax^2 + by^2 + cz^2$$

The *constraint 'hyperplane'* g is

$$g(x, y, z) = \sqrt{a}x + \sqrt{b}y + \sqrt{c}z - \sqrt{d} = 0$$

Step 2 The *Lagrange Multiplier* λ is so chosen that the function

$$\varphi = f + \lambda g = ax^2 + by^2 + cz^2 + \lambda(\sqrt{a}x + \sqrt{b}y + \sqrt{c}z - \sqrt{d})$$

satisfies

$$\frac{\partial}{\partial x}\varphi = 0\,, \frac{\partial}{\partial y}\varphi = 0\,, \frac{\partial}{\partial z}\varphi = 0 \;\&\; \frac{\partial}{\partial \lambda}\varphi = 0$$

$$2ax + \lambda\sqrt{a} = 0\,, 2by + \lambda\sqrt{b} = 0\,, 2cz + \lambda\sqrt{c} = 0 \;\&\; \sqrt{a}x + \sqrt{b}y + \sqrt{c}z - \sqrt{d} = 0$$

Since the 4th relation is already true, we use the first three

$$2ax = -\lambda\sqrt{a}\,, 2by = -\lambda\sqrt{b} \;\&\; 2cz = -\lambda\sqrt{c}$$

$$x = \sqrt{\frac{c}{a}}z \;\&\; y = \sqrt{\frac{c}{b}}z$$

Step 3 Putting these in the *constraint function* g we get

$$\sqrt{a}\left(\sqrt{\frac{c}{a}}z\right) + \sqrt{b}\left(\sqrt{\frac{c}{b}}z\right) + \sqrt{c}z - \sqrt{d} = 0$$

$$z = \frac{1}{3}\sqrt{\frac{d}{c}}\,, y = \sqrt{\frac{c}{b}} \times \frac{1}{3}\sqrt{\frac{d}{c}} \;\&\; x = \sqrt{\frac{c}{a}} \times \frac{1}{3}\sqrt{\frac{d}{c}}$$

$$z = \frac{1}{3}\sqrt{\frac{d}{c}}\,, y = \frac{1}{3}\sqrt{\frac{d}{b}} \;\&\; x = \frac{1}{3}\sqrt{\frac{d}{a}}$$

which means, as proved earlier, point $\left(\frac{1}{3}\sqrt{\frac{d}{a}}, \frac{1}{3}\sqrt{\frac{d}{b}}, \frac{1}{3}\sqrt{\frac{d}{c}}\right)$ is the *point of minimum*.

Example 12.2.2.5 What should be the dimensions of a parallelopiped to have maximum volume if it were to be made from a given sheet of metal of area $A = 6a^2$.

Solution:

Step 1 The '*volume' function* f to be maximized is

$$f(l, b, h) = lbh$$

Since the metal sheet is of area A we can write the *constraint* as $6a^2 = 2lb + 2bh + 2hl$. The '*constraint' function* g can therefore be defined as

$$g(l, b, h) = 2lb + 2bh + 2hl - 6a^2 = 0$$

Step 2 The *Lagrange Multiplier* λ is so chosen that the function

$$\varphi = f + \lambda g = lbh + \lambda(2lb + 2bh + 2hl - 6a^2)$$

satisfies

$$\frac{\partial}{\partial l}\varphi = 0\,, \frac{\partial}{\partial b}\varphi = 0\,, \frac{\partial}{\partial h}\varphi = 0 \;\&\; \frac{\partial}{\partial \lambda}\varphi = 0$$

$$bh + \lambda(2b + 2h) = 0\,, lh + \lambda(2l + 2h) = 0\,, lb + \lambda(2b + 2l) \;\&\; 2lb + 2bh + 2hl - 6a^2 = 0$$

Since the 4th relation is already true, we use the first three

$$\left(\frac{1}{h} + \frac{1}{b}\right) = -\frac{1}{2\lambda}\,, \left(\frac{1}{h} + \frac{1}{l}\right) = -\frac{1}{2\lambda}\,, \left(\frac{1}{l} + \frac{1}{b}\right) = -\frac{1}{2\lambda}$$

$$b = l \;\&\; h = b$$

$$l = b = h$$

Step 3 Putting these in the *constraint function* g we get

$$6l^2 = 6a^2$$

$$l = b = h = a$$

which means that the cylinder must be of the dimensions (a, a, a) with volume a^3.

Example 12.2.2.6 Find any local maxima or minima of the function $f = ax^{-2} + by^{-2} + cz^{-2}$ subject to the constraint $\sqrt{a}x + \sqrt{b}y + \sqrt{c}z - \sqrt{d} = 0$ using the *Lagrange Multiplier*.

Solution:

Step 1 The '*hypersurface' function* f to be minimized is

$$f(x, y, z) = a^4x^{-2} + b^4y^{-2} + c^4z^{-2}$$

The *constraint 'hyperplane'* g is

$$g(x, y) = ax + by + cz - d = 0$$

Step 2 The *Lagrange Multiplier* λ is so chosen that the function

$$\varphi = f + \lambda g = a^4x^{-2} + b^4y^{-2} + c^4z^{-2} + \lambda(ax + by + cz - d)$$

satisfies

$$\frac{\partial}{\partial x}\varphi = 0\,, \frac{\partial}{\partial y}\varphi = 0\,, \frac{\partial}{\partial z}\varphi = 0 \;\&\; \frac{\partial}{\partial \lambda}\varphi = 0$$

$$-2a^4x^{-3} + \lambda a = 0\,, -2b^4y^{-3} + \lambda b = 0\,, -2c^4z^{-3} + \lambda c = 0 \;\&\; ax + by + cz - d = 0$$

Since the 4th relation is already true, we use the first three

$$2a^4x^{-3} = \lambda a\,, 2b^4y^{-3} = \lambda b \;\&\; 2c^4z^{-3} = \lambda c$$

$$\left(\frac{x}{a}\right)^3 = \frac{2}{\lambda}\,, \left(\frac{y}{b}\right)^3 = \frac{2}{\lambda} \;\&\; \left(\frac{z}{c}\right)^3 = \frac{2}{\lambda}$$

$$x = \frac{a}{c}z \,\&\, y = \frac{b}{c}z$$

Step 3 Putting these in the *constraint function* g we get

$$a \times \frac{a}{c}z + b \times \frac{b}{c}z + cz - d = 0$$

$$z = \frac{c}{a^2 + b^2 + c^2}d\,, y = \frac{b}{a^2 + b^2 + c^2}d \,\&\, x = \frac{a}{a^2 + b^2 + c^2}d$$

which means point $\left(\frac{a}{a^2+b^2+c^2}d, \frac{b}{a^2+b^2+c^2}d, \frac{c}{a^2+b^2+c^2}d\right)$ is the *point of extrenum*.

Example 12.2.2.7 Find the maximum or minimum distance of the function $x^2 + y^2 = a^2$ from the point (h, k) using the *Lagrange Multiplier*.

Solution:

Step 1 since the distance of any point (x, y) on the curve from the point (h, k) is $\sqrt{(x-h)^2 + (y-k)^2}$, the '*distance' function* f to be extremized is defined as

$$f(x, y) = (x - h)^2 + (y - k)^2$$

The *constraint 'curve'* g is then

$$g(x, y) = x^2 + y^2 - a^2 = 0$$

Step 2 The *Lagrange Multiplier* λ is so chosen that the function

$$\varphi = f + \lambda g = (x - h)^2 + (y - k)^2 + \lambda(x^2 + y^2 - a^2)$$

satisfies

$$\frac{\partial}{\partial x}\varphi = 0\,, \frac{\partial}{\partial y}\varphi = 0 \,\&\, \frac{\partial}{\partial \lambda}\varphi = 0$$

$$2(x - h) + \lambda(2x) = 0\,, 2(y - k) + \lambda(2y) = 0 \,\&\, x^2 + y^2 - a^2 = 0$$

Since the 3rd relation is already true, we use the first two

$$x - h + \lambda x = 0 \,\&\, y - k + \lambda y = 0$$

$$(1 + \lambda)x = h \,\&\, (1 + \lambda)y = k$$

$$x = \frac{h}{(1 + \lambda)} \,\&\, y = \frac{k}{(1 + \lambda)}$$

Step 3 Putting these in the *constraint function* g we get

$$\left\{\frac{h}{(1+\lambda)}\right\}^2 + \left\{\frac{k}{(1+\lambda)}\right\}^2 - a^2 = 0$$

$$\frac{h^2}{(1+\lambda)^2} + \frac{k^2}{(1+\lambda)^2} = a^2$$

$$\frac{(h^2+k^2)}{a^2} = (1+\lambda)^2$$

$$1+\lambda = \pm\frac{\sqrt{(h^2+k^2)}}{a}$$

If we let $l \equiv \sqrt{(h^2+k^2)}$ then

$$1+\lambda = \pm\frac{l}{a}$$

which means points $\left(\frac{a}{l}h, \frac{a}{l}k\right)$ & $\left(-\frac{a}{l}h, -\frac{a}{l}k\right)$ is the *point of extrenum*.

Step 4 Putting these in the *distance* s we get

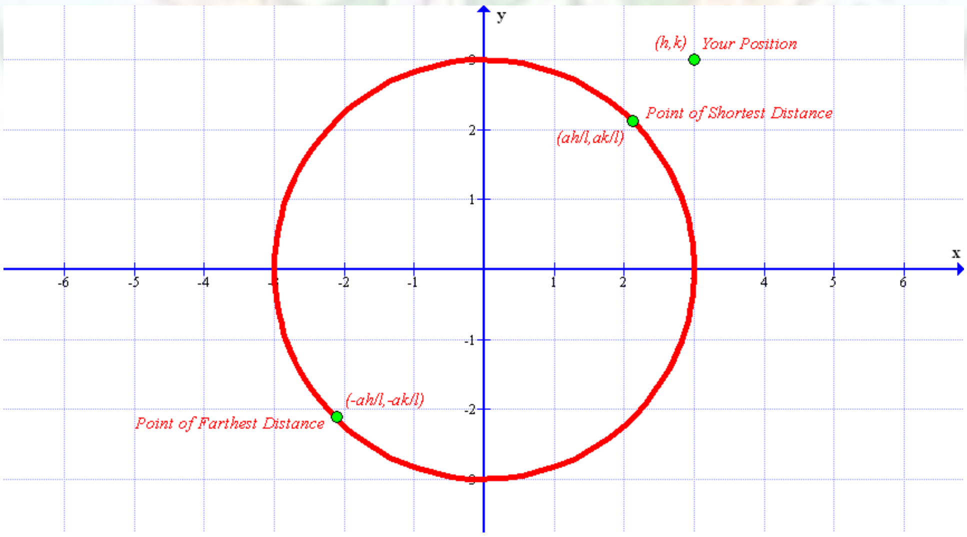
$$s(x1,y1) = \sqrt{\left(\frac{a}{l}h-h\right)^2 + \left(\frac{a}{l}k-k\right)^2} = \sqrt{\left(\frac{a}{l}-1\right)^2} \times l$$

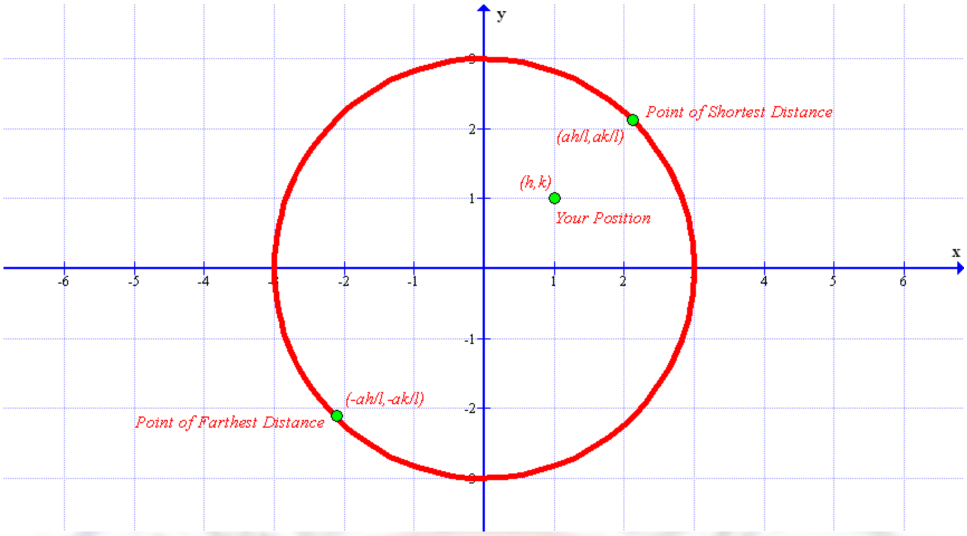
$$s(x1,y1) = a-l \quad if\ a > l$$

$$s(x1,y1) = l-a \quad if\ a < l$$

$$s(x2,y2) = \sqrt{\left(-\frac{a}{l}h-h\right)^2 + \left(-\frac{a}{l}k-k\right)^2} = \sqrt{\left(\frac{a}{l}+1\right)^2} \times l$$

$$s(x2,y2) = a+l \quad for\ any\ l$$





Summary

Maxima & Minima of a Function

- A function $f(x)$ represents a curve $y = f(x)$ in the (x, y) coordinate plane.
- There may exist point(s) (x_m, y_m) in the (x, y) coordinate plane which may be the lowest point (minimum) or the highest point (maximum) of the curve.
- These points could be identified by a condition that at these point(s) $\left.\frac{d}{dx}f\right|_{x=x_m} = 0$
- If at the point (x_m, y_m) $\left.\frac{d^2}{dx^2}f\right|_{x=x_m} > 0 \rightarrow$ *Point of Minimum* & $\left.\frac{d^2}{dx^2}f\right|_{x=x_m} < 0 \rightarrow$ *Point of Maximum*

Maxima & Minima of a Function under Constraints

- *Method of Elimination of a Variable*

  If (x_m, y_m) is the point on a curve in the (x, y) coordinate plane defined by a function $g(x)$ which is closest (or farthest) to a given point (x_0, y_0) it must satisfy the condition $\left.\frac{d}{dx}f\right|_{x=x_m} = 0$ where $f = \sqrt{[x - x_0]^2 + [g(x) - y_0]^2}$.

  To find whether it is the shortest distance we have to check for $\left.\frac{d^2}{dx^2}f\right|_{x=x_m} > 0$

- *Method of Lagrange Multiplier*

  If $f(x)$ is the *main function* and $g(x) = 0$ is the *constraint function*, then the *point of extremum* (x_m, y_m) of the *main function* can be found by imposing the two conditions $\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0$ & $\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0$ where λ is the *Lagrange Multiplier*.

  This method is even more powerful as it can be extended to two or more constraint functions. Hence, if $f(x, y)$ is the *main function* and $g_1(x, y) = 0$ & $g_2(x, y) = 0$ are two *constraint functions*, then the *point of extremum*

(x_m, y_m) of the *main function* can be found by defining a function $\varphi \equiv f(x,y) + \lambda_1 g_1(x,y) + \lambda_2 g_2(x,y)$ and demanding $\frac{\partial f}{\partial x} + \lambda_1 \frac{\partial g_1}{\partial x} + \lambda_2 \frac{\partial g_2}{\partial x} = 0$ & $\frac{\partial f}{\partial y} + \lambda_1 \frac{\partial g_1}{\partial y} + \lambda_2 \frac{\partial g_2}{\partial y} = 0$.

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