Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Rotational Dynamics / Mechanics-I

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Chapter: Rotational Dynamics

Rotational Dynamics / Mechanics-I

1.1 Introduction

This lesson is focused on developing the basic concepts like angular momentum of a particle, the torque exerted on the particle by the force about a point, its relationship with angular momentum, the condition on the conservation of angular momentum and the total angular momentum and the net torque exerted on a system of particles. This will enable us to see how Newton's laws of motion, hitherto studied for translational motion, can be generalized to objects executing rotational motion.

Objectives

After studying this lesson, you should be able to

- define the angular momentum of a particle moving uniformly in a straight line about a fixed point
- develop the concept of torque exerted on the particle by the external force about a point and relate it with the rate of change of its angular momentum
- generalize the definition of angular momentum and torque to three dimensions, and understand clearly the directions of (i) the angular momentum vector with respect to the position and the momentum vectors and (ii) the torque with respect to the force and the position vectors of a rotating object.
- learn the conservation of angular momentum of an object moving under the influence of a central force
- write the expressions for total angular momentum and net torque for a system of particles, each of which has an individual angular momentum and torque
- prove that the total kinetic energy of a system of particles can be expressed as sum of kinetic energy of centre of mass motion and kinetic energy of motion with respect to centre of mass of the system.

1.2 Angular Momentum of a Particle about a point

Let us begin with the simplest case of one particle with no forces acting on it. Consider a single particle P of mass m moving along a straight line AB distant d from the origin O with a uniform velocity **v**. Let **r** be the position of the particle represented by the position vector OP making angle θ with the velocity vector.

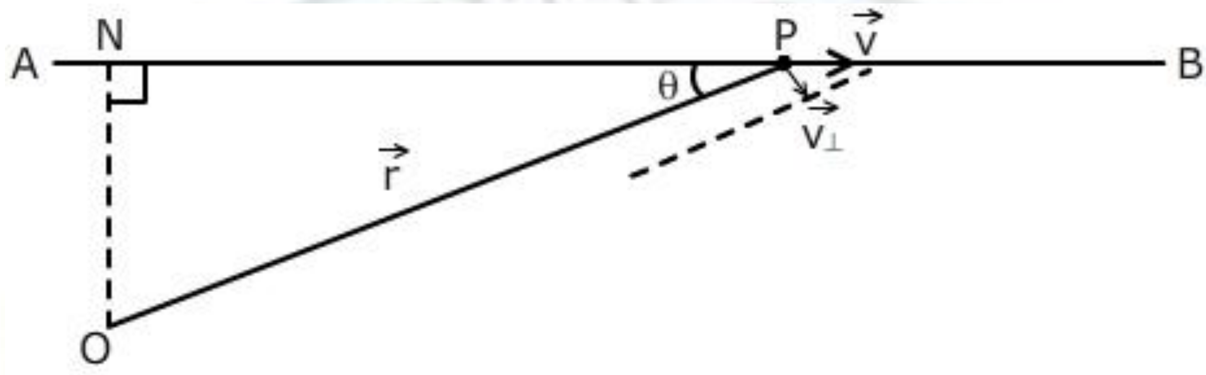


Fig.1.1 Uniform motion of a particle P in a straight line. The angular momentum L of the particle about the origin is m v r sin(θ)

The angular momentum L of the particle P about the origin O is defined to be the perpendicular distance ON from O to the line AB and the momentum of the particle, m v. Since ON=OP $\sin\theta$ =r $\sin\theta$, we can symbolically write angular momentum

$$L = m\,v\,r\sin\theta \qquad (1.1)$$

Alternatively, since the velocity component perpendicular to the line OP i.e., to the radius vector r is v $\sin\theta \equiv v_\perp$ and we know that the angular speed of the particle, $\omega = v_\perp / r$, we can express angular momentum

$$L = m\,r\,v_\perp = mr^2\omega \qquad (1.2)$$

Adopting the sign convention, L is positive if the line OP turns in the positive sense(i.e., anticlockwise) in the plane. In the present example, shown in the figure, L is negative.

Suppose that in time Δt , the particle P moves a distance vΔt from point P to P_1, then the area swept out in a time Δt is (see the figure)

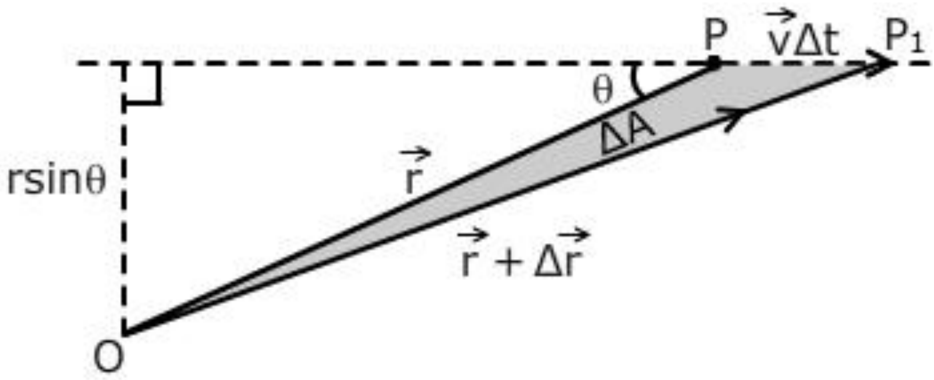


Fig. 1.2 The area $\Delta A = OPP_1$ is swept out in a time Δt. The ratio $\Delta A/\Delta t$ is called the areal velocity.

$$\Delta A = \frac{1}{2} v\Delta t \times r\sin\theta \qquad (1.3)$$

The area swept out per unit time is called the areal velocity, $\Delta A/\Delta t$, which is

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} v \times r\sin\theta \qquad (1.4)$$

From Eq.(1.1), we note that angular momentum per unit mss, L/m, is just v r sin θ.

On comparing it with Eq.(1.4), we see that areal velocity is just half of the angular momentum per unit mass. Further, as the free particle continues to move along the line AB and the perpendicular distance ON from the origin does not change with time, the angular momentum of the particle about the origin remains constant.

In order to generalize the definition in three dimensions, angular momentum is regarded as a vector quantity $\vec{L}$. **It is defined as the vector product of vectors** $\vec{r}$ **and** $\vec{p}$.

$$\vec{L} = \vec{r} \times \vec{p} \qquad (1.5)$$

The magnitude of angular momentum vector is the same as given by Eq.(1.1) and its direction is perpendicular to the plane containing the vectors r and p.

You can directly verify that L is constant by calculating the change $\Delta\vec{L}$ of the vector $\vec{L}$ after a short interval of time Δt. Thus

$$\begin{aligned}\Delta\vec{L} &= \vec{L}(t+\Delta t) - \vec{L}(t)\\ &= \vec{r}(t+\Delta t)\times\vec{p} - \vec{r}\times\vec{p} = [\vec{r}(t+\Delta t) - \vec{r}(t)]\times\vec{p}\end{aligned} \qquad (1.6)$$

Note that it is only vector $\vec{r}$ which is changing as t changes to t+Δt, whereas the vector $\vec{p}$ remains constant. The rate of change of angular momentum is therefore given by

$$\frac{\Delta \vec{L}}{\Delta t} = \frac{\vec{v}\Delta t \times \vec{p}}{\Delta t} = \vec{v} \times \vec{p} = 0 , \tag{1.7}$$

since vectors $\vec{v}$ and $\vec{p}$ are parallel to each other.

Exercise: Show that, by a suitable choice of the origin, angular momentum of a free particle can be made zero.

Solution

Move the origin O on the line of the motion of the particle. With this choice, the momentum vector **p** is parallel to the position vector **r**, so that

$$\vec{L} = \vec{r} \times \vec{p} = 0$$

1.3 Torque

Let us introduce the concept of torque by considering that particle P now experiences a force F. We can again calculate the change of angular momentum ΔL in a small time interval Δt. The displacement in this small time is vΔt. However, in this small interval, the momentum also changes from $\vec{p}$ to $\vec{p} + \Delta\vec{p}$, where, let us recall from Newton's second law of motion, $\Delta\vec{p} / \Delta t = \vec{F}$. Keeping this in mind, the change of angular momentum is

$$\Delta\vec{L} = (\vec{r} + \Delta\vec{r}) \times (\vec{p} + \Delta\vec{p}) - \vec{r} \times \vec{p} = \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p} + \Delta\vec{r} \times \Delta\vec{p} \tag{1.8}$$

The contribution of the term, $\Delta\vec{r} \times \vec{p}$, is zero since, as before, the velocity is parallel to p. We would thus get

$$\Delta\vec{L} = (\vec{r} \times \vec{F})\Delta t + \Delta\vec{r} \times \Delta\vec{p} \tag{1.9}$$

Therefore, the rate of change of angular momentum is

$$\frac{\Delta\vec{L}}{\Delta t} = \vec{r} \times \vec{F} + \frac{\Delta\vec{r}}{\Delta t} \times \vec{F}\Delta t \tag{1.10}$$

Taking the limit $\Delta t \to 0$, the second term tends to zero. We get

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{T} \tag{1.11}$$

The vector $\vec{r} \times \vec{F}$, is, by definition, the torque denoted by the vector $\vec{T}$ (it is frequently denoted by the symbol $\vec{\tau}$ also), exerted on the particle by the force $\vec{F}$ about the origin.

Physically speaking, torque is a measure of the turning force to rotate an object about an axis. The most common example from everyday life is when you try to open the door. Consider, for example, a door hinged at some point O (see Fig.1.3) which is free to rotate about a line perpendicular to the plane of the page.

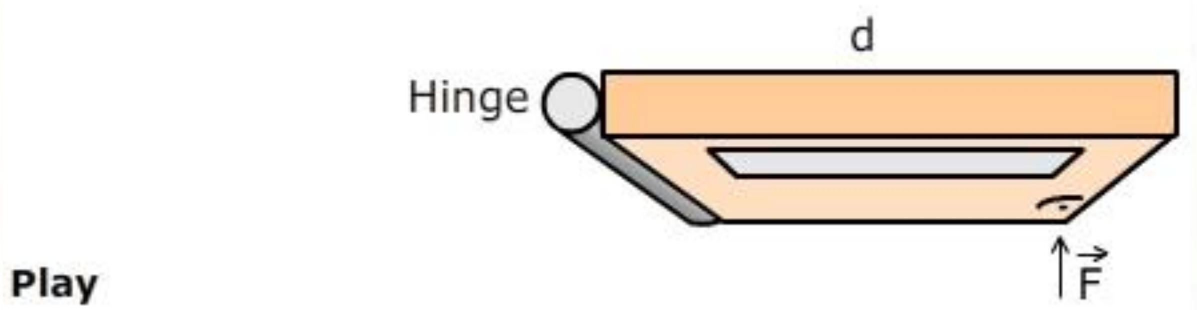


Figure 1.3 : An overhead view of a door hinged at point O with a force F applied perpendicular to the door.

When the force F is applied at the outer edge, as shown, the door can easily rotate anticlockwise. The effect on rotation is quite large as compared to a situation when the same force is applied at the point near the hinge. The magnitude of the torque T=F d, where the distance d is the lever arm of the force F. It is the perpendicular distance from the axis of rotation to the line joining the direction of the force. For a given force, greater the distance d, greater would be the torque. Torque is a vector perpendicular to the plane determined by the lever arm and the force. Its value depends on the axis of rotation.

Exercise

If the torque required to loosen a nut that is holding a flat tyre in place on a car has magnitude of 30.0 N-m, what minimum force must be exerted by the mechanic at the end of a 30.0 cm long wrench to accomplish this task? What would be the force required if he had a 20 cm long wrench?

Answer:

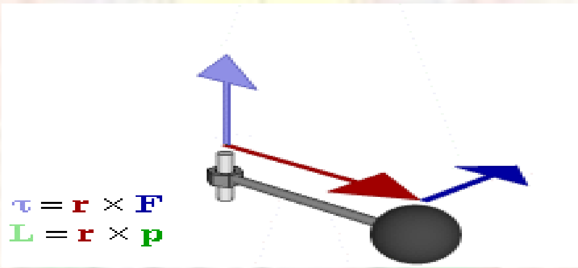
Torque, T= r×F=30.0 N-m

Since, r=30 cm, F=30/0.3=100 N.

For r=20 cm, F=30/0.20=150 N .

Value Addition:

You are advised to visit the following website for an animation giving you a feeling how a force(F) acting on a rotating body with position vector(r) gives rise to the angular momentum(L) and the torque(T). Watch carefully how during rotation, the directions of angular momentum and torque are changing with respect to the directions of the force



and the position vector r.

http://en.wikipedia.org/wiki/File:Torque_animation.gif#file

Licensing [edit]

Summary [edit]

Description	Animated GIF image demonstrating relationship between force (F), torque (τ), linear momentum (p), angular momentum (L), and position (r) of rotating particle.
Date	21 February 2008
Source	Own work
Author	Yawe

There are certain situations when $\vec{F}$ is parallel to vector $\vec{r}$, i.e., the force acts towards (or away from) the centre. The force is then said to be a **'central'** force. The common examples are the gravitational force between two masses or the Coulomb force between two charged particles. In such cases, torque would be zero and therefore

$d\vec{L}/dt = 0$, implying thereby that **the angular momentum of the particle is conserved**.

Exercise

Show that the angular momentum of a particle, moving under central force is conserved.

Proof:

A central force acting on a particle depends only upon the magnitude of its distance from a fixed centre. If $\vec{r}$ is the instantaneous position vector of the particle relative to the fixed centre O, then the central force is represented by

$$\vec{F} = f(r)\hat{r} \quad , \tag{1.12}$$

where f(r) is a scalar function of distance r and $\hat{r} = \vec{r}/r$.

The torque acting on the particle is

$$\vec{T} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r} = f(r)\left[\vec{r} \times \frac{\vec{r}}{r}\right] = 0 \tag{1.13}$$

So that $$\vec{L} = \vec{r} \times m\,\vec{v} = constant. \tag{1.14}$$

If the angular momentum, $\vec{L}$, is constant, it should be perpendicular to the plane containing both $\vec{r}$ *and* $\vec{v}$. This implies that the path of the particle under the influence of central force lies in the plane. As discussed above (cf., Fig.(1.2)), when the vector $\vec{r}$ from the centre of force O changes to $\vec{r} + \Delta\vec{r}$, the vector area swept by the radius vector during time interval Δt is given by

$$\Delta\vec{A} = (\frac{1}{2}\vec{r} \times \Delta\vec{r})\Delta t. \tag{1.15}$$

Therefore, the areal velocity

$$\Delta\vec{A}/\Delta t = (\frac{1}{2}\vec{r} \times \Delta\vec{r}) = \vec{L}/2m \tag{1.16}$$

Since angular momentum $\vec{L}$ is constant for central forces, this shows that areal velocity remains constant, when the particle moves under the influence of central force.

1.4 Angular Momentum, Torque and Kinetic Energy for a System of Particles

Let us consider a system of particles with both external and internal forces acting on them. The total angular momentum of the system can be obtained by adding the angular momenta of the individual particles, i.e.,

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 + \ldots\ldots\ldots\ldots\ldots\ldots + \vec{r}_n \times \vec{p}_n \qquad (1.17)$$

Similarly the total external torque acting on the system is given by

$$\vec{T} = \vec{r}_1 \times \vec{F}_{1e} + \vec{r}_2 \times \vec{F}_{2e} + \ldots\ldots\ldots\ldots\ldots\ldots + \vec{r}_n \times \vec{F}_{ne} \qquad (1.18)$$

This result is clearly the generalization of Eq.(1.11) obtained above to a system of particles. Another important generalization is that the total torque exerted by external forces is the rate of change of the total angular momentum, which can be mathematically expressed as

$$\vec{T}_e = \frac{d\vec{L}}{dt}$$

Indeed this equation assumes that internal forces do not contribute to the change of the total angular momentum. This is in accordance with common experience, viz., bodies do not spin on their own without external torques acting on them. **This equation is clearly the rotational analogue of the equation**

$$\vec{F}_e = \frac{d\,\vec{p}}{d\,t}\,.$$

Let us now consider the kinetic energy of a system of particles. Assume that the position vector of a particle of mass m_i (i=1 to n) is $\vec{r}_i$ with respect to the centre of mass of the system. Then if the position vector of the centre of mass from the origin is $\vec{R}$, the position vector of the ith particle with respect to the origin is $\vec{R}+\vec{r}_i$. Thus the kinetic energy, K, of this mass is

$$K = \frac{1}{2} m_i \left(\frac{d\vec{R}}{dt} + \frac{d\vec{r}_i}{dt} \right)^2$$

$$= \frac{1}{2} m_i \left(\frac{d\vec{R}}{dt} \right)^2 + \frac{1}{2} m_i \left(\frac{d\vec{r}_i}{dt} \right)^2 + m_i \frac{d\vec{R}}{dt} . \frac{d\vec{r}_i}{dt} \qquad (1.19)$$

Adding up the contributions of all the particles, we find the first term becomes

$$\frac{1}{2} \left(\sum m_i \right) \left(\frac{d\vec{R}}{dt} \right)^2 \ , \qquad (1.20)$$

which represents the kinetic energy of the motion of centre of mass.

The term $d\vec{r}_i / dt$ is the velocity of the ith particle in the frame in which the centre of mass is at rest. Thus the second term contributes to the kinetic energy of the motion with respect to the centre of mass, which is

$$\frac{1}{2}\sum m_i \left(\frac{d\vec{r}_i}{dt}\right)^2 \qquad (1.21)$$

So far we obtain the sum of two kinetic energies. But, what is the contribution of the third term? The third term, summed over all the particles, can be expressed as

$$\left(\frac{d\vec{R}}{dt}\right)\sum \frac{d}{dt}\left(\frac{m_i \vec{r}_i}{M}\right) M$$

Look at the expression $\sum (m_i \vec{r}_i / M)$. This represents the position vector $\vec{R}$ of the centre of mass with respect to itself, i.e., zero. Therefore, the third term vanishes.

We have thus obtained the basic result, viz.,

Total kinetic energy of a system of particles= Kinetic Energy of CM motion + kinetic energy of motion with respect to CM.

Summary

In this lesson you study

- the basic concepts of (i) angular momentum of a particle moving with uniform velocity about a point ; (ii) the torque exerted on the particle by a force and its relationship with angular momentum
- the generalization of these physical quantities to write the relations in vector form
- conservation of angular momentum of an object moving under the influence of a central force
- the expressions for total angular momentum and net torque for a system of particles, each of which has an individual angular momentum and torque
- that the total kinetic energy of a system of particles can be expressed as a sum of kinetic energy of centre of mass motion and kinetic energy of motion with respect to centre of mass of the system.

Exercises (for practice) :

Q. 1 The angular momentum and torque acting on the objects about their respective points are vectors which are respectively related to the linear momentum vector and the force vector. In what way are these vectors different from the momentum and force vectors?

Answer: The linear momentum $\vec{p}$ and the force $\vec{F}$ are known as polar vectors, which change sign under inversion, whereas angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is an axial vector which does not change sign under inversion (since both the vectors, $\vec{r}$ *and* $\vec{p}$ change sign under inversion). Similarly, the torque, $\vec{T} = \vec{r} \times \vec{F}$ is also an axial vector . Both the angular momentum and the torque cause rotational motion whereas linear momentum and force refer to the translational motion.

Q. 2 The torque acting on a body about a given point is expressed as $\vec{T} = \vec{A} \times \vec{L}$, where $\vec{A}$ is a constant vector and $\vec{L}$ is angular momentum of the body about the point. From the statements given below, tick the one which is **true / false**, giving reason:

(a) $\frac{d\vec{L}}{dt}$ is perpendicular to $\vec{L}$ at all instants of time;

(b) The component of $\vec{L}$ in the direction of $\vec{A}$ changes with time.

Solution: Since torque is defined as $\frac{d\vec{L}}{dt}$ and it is expressed as a cross product of the vectors, $\vec{A}$ and $\vec{L}$, it must, by definition, be perpendicular to $\vec{L}$. Statement (a) is true. Since $\frac{d\vec{L}}{dt}$ is perpendicular to the vector $\vec{A}$, the component of $\vec{L}$ in the direction of $\vec{A}$ can not change with time, the statement (b) is false.

Q. 3 In the question given below, mark the correct choice, justifying your answer.

A mass is moving with a constant velocity along a line parallel to the x-axis away from the origin. The angular momentum with respect to the origin

(a) is zero (b) remains constant (c) goes on increasing

(d) goes on decreasing

Answer : The correct choice is (b). The reason is, mass is moving with constant velocity and the perpendicular distance from the origin to the x-axis remains constant, since particle is only moving along the x-axis.

Q. 4 A particle of mass m is whirled in a circular path with constant angular velocity and its angular momentum is L. If the string is now halved, keeping the angular velocity same, how would its angular momentum be affected?

Solution: In a circular orbit, the angular momentum L=r p= mv r. Since v=rω, L=m $r^2\omega$, and if the string is now halved, i.e., it becomes r/2, therefore the angular momentum is reduced to L/4.

Q. 5 Find out the magnitude and direction of the torque, about the origin, due to a force, $\vec{F} = F_0\,\hat{k}$ newton, acting on a point whose position vector $\vec{r} = 5i + 5\,\hat{j}$ meter. Is the torque also perpendicular to r?

Answer: The torque $\vec{T} = \vec{r} \times \vec{F}$. Substituting the expressions for the position vector and the force, we get $\vec{T} = 5(\hat{i} - \hat{j})$. Its magnitude is $5\sqrt{2}\, n - m$, and it acts in the x-y plane, making the angle θ=-π/4 with the x-axis. Yes, the vector T is also perpendicular to the vector r (scalar product is zero).

Q. 6 Mark the correct choice, justifying your answer

The angular momentum of a particle moving in a circular orbit with a constant speed remains constant about

(a) any point on the circumference of the circle
(b) any point inside the circle
(c) any point out of the circle
(d) the centre of the circle.

Answer: In a circular orbit, the radius of the orbit is constant and also since the particle has the same speed, L =m v r remains constant. At any point inside or outside the circle, the distance and angle of the orbiting particle with respect to the point would be changing with time. And at any point on the circumference, r would be zero. The correct choice is, therefore, (d).

Q. 7 The position vector of a particle with respect to origin O is $\vec{r}$. If the torque acting on the particle is zero, out of the following statements, mark the ones which are correct:

(a) Linear momentum of the particle remains constant
(b) Angular momentum of the particle about O is constant
(c) The force applied to the particle is perpendicular to $\vec{r}$
(d) The force applied to the particle is parallel to $\vec{r}$.

Answer. Since torque is zero, it means $\frac{d\vec{L}}{dt}$=0. So $\vec{L}$ is constant. Statement (b) is therefore correct. Also, if torque, which is given by, $\vec{T} = \vec{r} \times \vec{F}$, if zero would imply that the force is parallel to r. So statement (d) is also correct.

Q. 8 A particle moves in a circular orbit with uniform angular speed. However, the plane of the circular orbit is itself rotating at a constant angular speed. Out of the following statements given below, mark the one which is **true** and which is **false**:

(a) The angular velocity of the particle remains constant but its angular acceleration varies
(b) The angular velocity of the particle varies but its angular acceleration is constant.

Answer: The angular velocity vector, being normal to the orbit, is constantly changing its direction. So statement (a) is false. But rate of change of this vector is constant, implying that angular acceleration is constant. Thus statement (b) is true.

Q. 9 Find out the centripetal force acting on a particle of mass m rotating in a plane in circular path of radius r and angular momentum L.

Answer: The angular momentum of the particle L=r p=m v r , which gives v= L/ m r.

Now, the centripetal force is $\frac{mv^2}{r} = \frac{L^2}{mr^3}$.

Q. 10 A stone tied to the end of a string of length l is whirled along a circular path. If the string suddenly breaks, what would happen?

(a) Would angular momentum of the body become zero?
(b) Would the stone drop down?
(c) Would the stone fly along the tangent to the circular path?
(d) Would the stone fly outward?

Answer: Since no external torque acts on the stone even after the string breaks, the angular momentum will remain unchanged (only centripetal force is no longer provided when the string breaks). The stone would fly along the tangent to the circular path. The correct choice is (c).

Q. 11 A comet is orbiting around the sun . The maximum and minimum distances of the comet from the sun are $1.4\times10^{12} m$ *and* $6\times10^{10} m$ respectively. If the velocity nearest to the sun is $7\times10^4 m/s$, what would be the velocity in the farthest position.

Solution: The area velocity of the comet is comet is constant, which implies

$$v_1 \ r_1 = v_2 \ r_2 \quad \Rightarrow \quad v_2 = \frac{v_1 \ r_1}{r_2} = \frac{6\times10^{10} \times 7\times10^4}{1.4\times10^{12}} = 3\times10^3 m/s$$

Q. 12 A disc is rotating with angular velocity $\vec{\omega}$. A force acts on a point whose position vector with respect to the axis of rotation is $\vec{r}$. Find the expression for the power associated with the torque due to the force.

Answer: Torque is given by $\vec{T} = \vec{r} \times \vec{F}$. The power associated with the torque

$$P = \vec{T}.\,\vec{\omega} = (\vec{r} \times \vec{F}).\,\vec{\omega}$$