Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Rotational Dynamics / Mechanics-II

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Rotational Dynamics / Mechanics-II

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Rotational Dynamics

Lesson 2

2.1 Introduction

In this lesson, we turn our attention to study the mechanics of a rigid body. A rigid body is one which maintains its shape even in the presence of external forces which can cause a translational or a rotational motion. In the presence of external forces acting on a body, it would be pertinent to understand the role played by the net torque acting on it to obtain the conditions for rotational equilibrium. An understanding of equilibrium problems is important in a variety of fields such as in architecture or civil engineering or even in biology to understand the forces working in muscles and joints.

Objectives

After studying this lesson, you should be able to

- state the requirements to be imposed on an object to be in mechanical equilibrium (including rotational equilibrium)
- define the moment of the force and use the principle of moments to locate the centre of gravity of an object
- establish a relationship between torque and the angular acceleration of a particle of mass m revolving about a radius r in terms of its moment of inertia
- extend the above relation in the case of a solid disc rotating about a given axis

2.2 Rotational Equilibrium and the Principle of Moments

We are all familiar with the common balance which simply consists of a beam turning about a fixed point. Two objects, because of their weights, exert downward forces on the beam at equal distances from the fixed point. When the two forces are equal, the balance does not turn in either direction. It is said to be in equilibrium. What happens when the two distances are not equal? We have learnt from elementary courses that when the weights W_1 and W_2 of the two bodies are at distances d_1 and d_2 from the point of rotation, there is equilibrium, [i.e., no rotation about the fulcrum(i.e, the point about which the beam is free to rotate)] when the two bodies satisfy the condition (sometimes called the lever principle)

$$W_1 d_1 = W_2 d_2 \quad . \qquad (2.1)$$

Let us try to generalize this result by considering a general case. Suppose there are two forces acting on an object as shown in the figure. In this example, force $\vec{F}1$ tends to rotate the object counter clockwise, whereas the force $\vec{F}2$ rotates it clockwise. <u>We use the convention that the sign of torque is positive if it has the tendency to turn counter clockwise and negative if it is turning clockwise.</u> Remember that the units of torque are units of force times length, i.e., newton. meter(N.m).

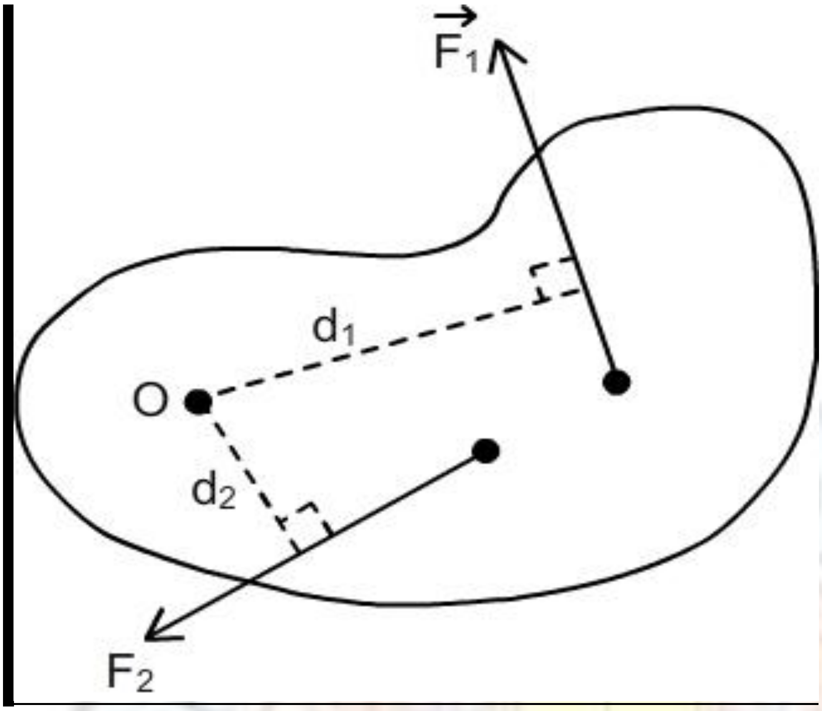


Fig. 2.1 The force $\vec{F}_1$ tends to rotate the object counter clockwise about O and $\vec{F}_2$ tends to rotate it clockwise.

From the figure, it is clear that the torque associated with the force $\vec{F}_1$, which has a moment arm d_1 is positive and equal to $F_1 d_1$; and the torque associated with $\vec{F}_2$ is negative and equal to $-F_2 d_2$. Thus the net torque acting on the object O is found by summing the torques:

$$\sum T = T_1 + T_2 = F_1 d_1 - F_2 d_2 \tag{2.2}$$

The product of the applied force, say, F_1 and the corresponding perpendicular distance, d_1, of its line of action from the point about which the body is free to rotate is also known as the **moment of the force.**

From the study of translational motion you have learnt that objects that are either at rest or moving with constant velocity are said to be in equilibrium. Since acceleration is zero, this condition is mathematically expressed as

$$\sum \vec{F} = 0 . \tag{2.3}$$

It means that the vector sum of all the forces (the net force) acting on an object in equilibrium is zero. Is this condition sufficient to ensure complete mechanical equilibrium? To answer this question, let us consider the following exercise:

Suppose we have a packing crate being pushed by two forces of equal magnitude but acting in opposite directions as shown in the figure (2.2).

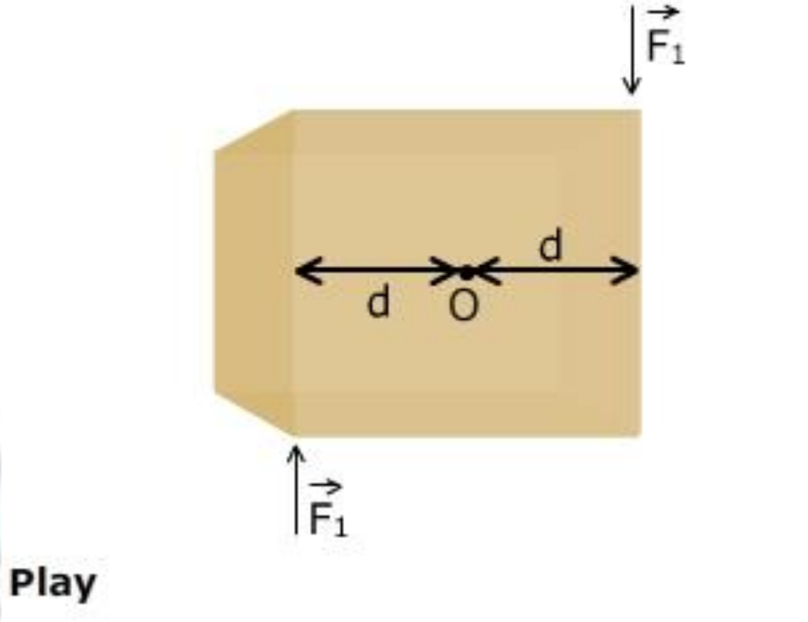


Fig. 2.2 A top view of a packing crate being pushed by forces of equal magnitude and in opposite directions

Here each force produces clockwise rotation, the resulting torques are both negative. The net torque produced by the two forces is -2Fd , i,e., producing clockwise rotation.

The example considered above illustrates that in order to understand the effect of a force or two or more forces on an object, we must know not only the magnitude and direction of the forces but also their points of application. In other words net torque acting on an object must also be considered. We are thus led to two requirements that must be imposed on an object to be in mechanical equilibrium:

1. The net external force must be zero. $\sum \vec{F} = 0$.
2. The net external torque must be zero. $\sum \vec{T} = 0$.

The first condition is obviously a statement of translational equilibrium, while the second is for rotational equilibrium.

One of the forces that must be considered while dealing with a rigid object is that of gravity acting on the object. To find out the torque due to force of gravity, all of the weight can be thought of concentrated at a single point.

Let us now apply the principle of moments by considering an object of arbitrary shape lying in the x-y plane as shown in the figure (Fig.2.3). Suppose that the object is divided into a large number of very small particles of masses m_1, m_2, etc., located at positions ($x_1, y_1), (x_2, y_2), (x_3, y_3)$............, etc., with reference to an origin O. If the object is free to rotate about the origin, each particle contributes a torque about the origin, which would be equal to its weight multiplied by its lever arm. For example, the torque due to the weight $m_1 g$ is $m_1 g x_1$ and so on.

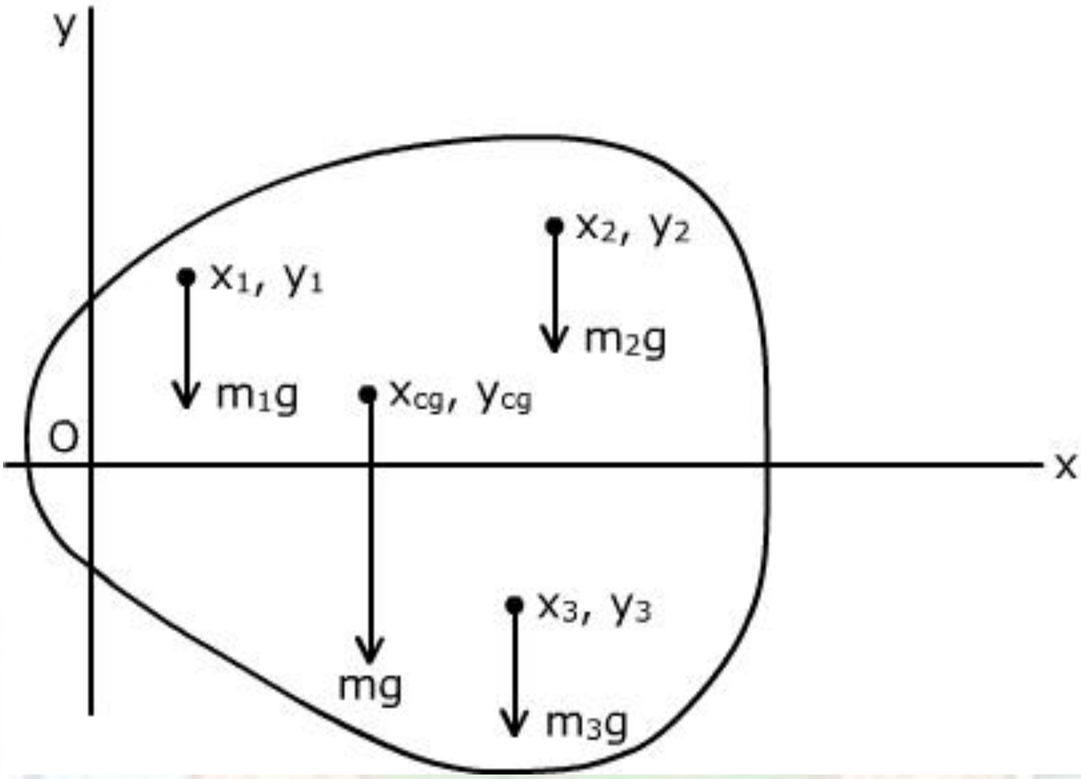


Fig. 2.3 The centre of gravity of an object where all the weight of the object can be considered to be concentrated.

The point where all the weight w of the object can be considered to be concentrated is called the **centre of gravity** of the object. This is the point of application of the single force whose effect on the rotation of the object is the same as that of the individual particles. To locate this point, we apply the principle of moments, or equivalently equate the torque exerted by w at the centre of gravity to the sum of the torques acting on the individual particles.

As the object is in equilibrium about the fulcrum, equating the torque exerted by w at the centre of gravity to the sum of the torque acting on the individual particles about the fulcrum must be zero. We get

$$m_1 g x_1 + m_2 g x_2 + m_3 g . x_3 + ... = (m_1 g + m_2 g + m_3 g + ...) x_{cg} \, , \tag{2.4}$$

which gives

$$x_{cg} \sum m_i = \sum m_i x_i \quad \text{or} \quad x_{cg} = \frac{\sum m_i x_i}{\sum m_i} \, . \tag{2.5}$$

Similarly, the y-coordinate of the centre of gravity of the system can be obtained as

$$y_{cg} = \frac{\sum m_i y_i}{\sum m_i} \tag{2.6}$$

Note that centre of gravity of a symmetric, homogeneous body always lies on the axis of symmetry. For example, centre of gravity of a homogeneous sphere or that of cube must lie at the geometric centre of the object. A homogeneous rod has its centre of gravity at its centre.

Example:

Find the centre of gravity of a triangle made up of three equal masses at its vertices as given in the figure (Fig.2.4).

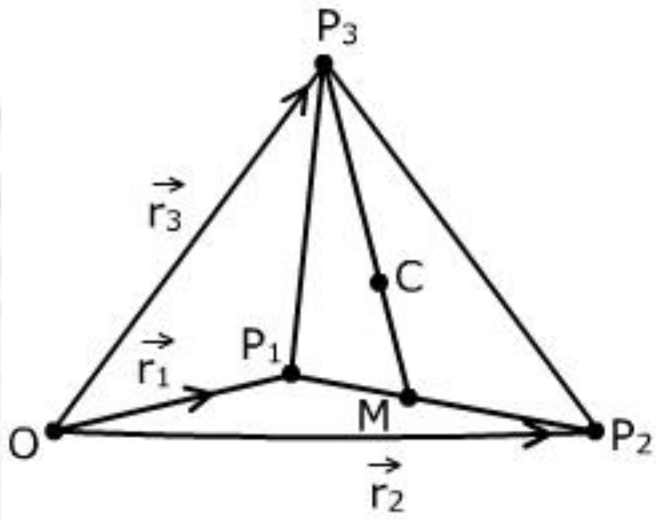


Fig. 2.4 Three equal masses placed at the vertices $P_1, P_2.P_3$ of a triangle.

Answer:

Let us write the two equations (2.5) and (2.6) given above in the vector form. Then

$$R_C = \frac{\sum m_i r_i}{\sum m_i}$$

In the case of the given triangular arrangement of masses, we write

$$R_C = \frac{m(r_1 + r_2 + r_3)}{3m}$$

$$= \left[\frac{\frac{2(r_1 + r_2)}{2} + r_3}{2+1} \right]$$

The last expression has the following physical meaning: The first term in the numerator of the bracket represents the centre of gravity of one particle of mass=2m, located at the mid point M of two masses at the vertices (1) and (2) and then combined with the third mass. The centre of gravity is thus obtained by taking the median of the triangle (line joining the vertex to the mid point of the opposite side) and dividing in the ratio of 2:1(see the figure**). It is interesting to notice that you could have started with any one of the three pairs and got the same result**. Do you know, Why? Because remember, medians of the triangle intersect at a common point, which divides all of them in the ratio o 2:1. **This is called the 'centroid'**. Even if we had considered the triangle made up of three uniform rods, the centre of gravity would be the same point.

Exercise

A uniform horizontal beam having weight of 500 N and 5.0 m long is attached to a wall through a 'plug' connection that allows the beam to rotate. Its far end is supported by a cable that makes an angle of 60^0 with the horizontal (See Fig. 2.5). If a person whose weight is 600 N stands on the beam at 1.5 m from the wall, find the tension in the cable and the force exerted by the wall on the beam.

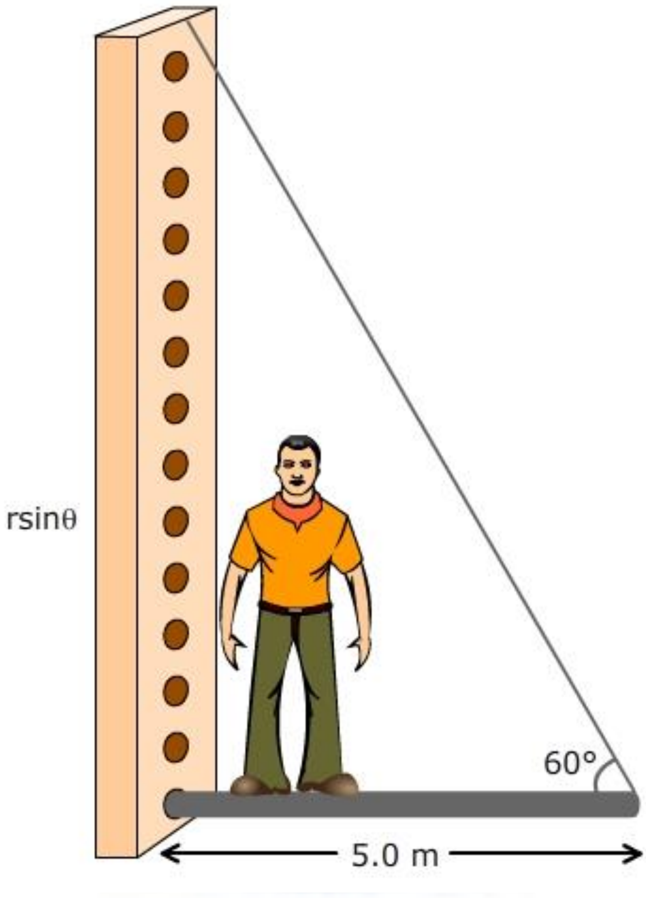
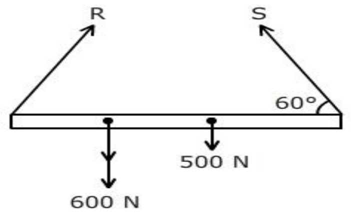


Fig. 2.5

Solution

Let us first identify the forces acting on the beam: (i) the forces on the beam consist of the downward force of gravity having a magnitude of 500 N at its centre of gravity; (ii) the downward force exerted by the man, which is his weight of 600 N acting downward at 1.5 m from the wall; the force of tension S exerted by the cable and the force R exerted by the wall.



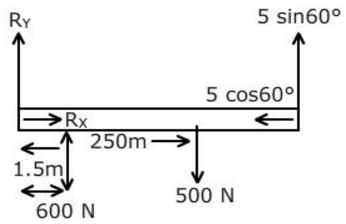
Now, applying the conditions for equilibrium on the x- and y-components of the forces,

we find

$$R_x - S\cos(60^0) = 0$$

$$R_y + S\sin(60^0) - 500N - 600N = 0$$

Clearly, there are three unknowns and only two equations. We have now to use the second condition of equilibrium, i.e., on the torque acting on the beam. Thus



$$(S\sin(60^0))(5.0m) - (500N)(2.5m) - (600N)(1.5m) = 0,$$

which enables us to get the value of S=500N. Using the above two equations, we get

$$R_x = 250N \qquad and \qquad R_y = 670N$$

2.3 Relationship between Torque and Angular Acceleration

Suppose we have a system consisting of an object of mass m connected to a very light rod of length l. The rod, pivoted at the point O, is rotating on a frictionless horizontal table as shown in Fig.2.6. Let us assume that a force $\vec{F_t}$ perpendicular to the rod and therefore tangential to the circular orbit is acting on mass m.

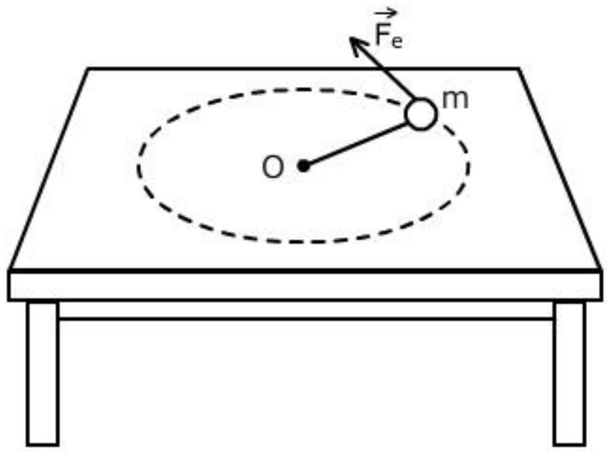


Fig.2.6

Due to this force the object undergoes a tangential acceleration given by

$$\vec{F_t} = m\vec{a_t} \quad , \qquad (2.7)$$

according to Newton's second law. Multiplying the left and right sides of this equation by r, we write the above equation as

$$\vec{F_t}\, r = m\vec{a_t}\, r \qquad (2.8)$$

From our earlier study, we know that tangential acceleration and angular acceleration $\vec{\alpha}$ of a particle rotating in a circular path are related by $\vec{a_t} = r\vec{\alpha}$. Thus, we have

$$\vec{F_t}\, r = mr^2\, \vec{\alpha} \qquad (2.9)$$

The left hand side of this equation is the torque acting on the object about its axis of rotation This gives us an important relation

$$\vec{T} = mr^2\vec{\alpha} \qquad (2.10)$$

showing that torque acting on the system is directly proportional to the angular acceleration. The constant of proportionality, $m\,r^2$, is called the **moment of inertia** of the object of mass m (since, by assumption, rod is considered light, its moment of inertia can be neglected).

2.5.1 Torque on a Rotating Object

Let us now extend our study to a rigid body. **A rigid body, as mentioned before, is a system of particles in which the relative positions of the particles remain fixed under the application of forces. It means that a rigid body retains its shape during motion.**

Consider a sold disc rotating about its axis as shown in Figure 2.7(a). The disc consists of many particles at various distances from the axis of rotation. This is illustrated in Fig.2.7(b). The torque acting on each one of these particles is given by

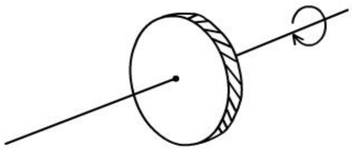


Fig. 2.7 (a)

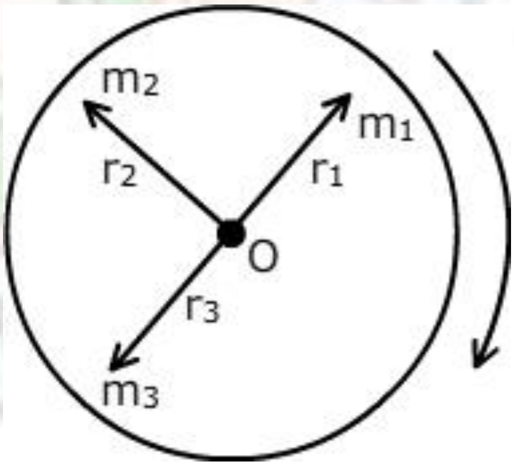


Fig.2.7(b)

Eq.(2.10). The total torque acting on the disc is the sum of the individual torques on all the particles, viz.,

$$\sum \vec{T} = \left(\sum m r^2\right)\vec{\alpha} \qquad (2.11)$$

Since the disc is rigid, all particles have the same angular acceleration, so $\vec{\alpha}$ is not appearing within the summation sign. Labelling the masses as m_1, m_2, m_3 .....located respectively at the positions r_1, r_2, r_3...... from the centre, as shown in Fig.2.7(b), we write

$$\sum m r^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ... \qquad (2.12)$$

This quantity is the moment of inertia of the disc and is given by the symbol I:

$$I = \sum m r^2 \qquad (2.13)$$

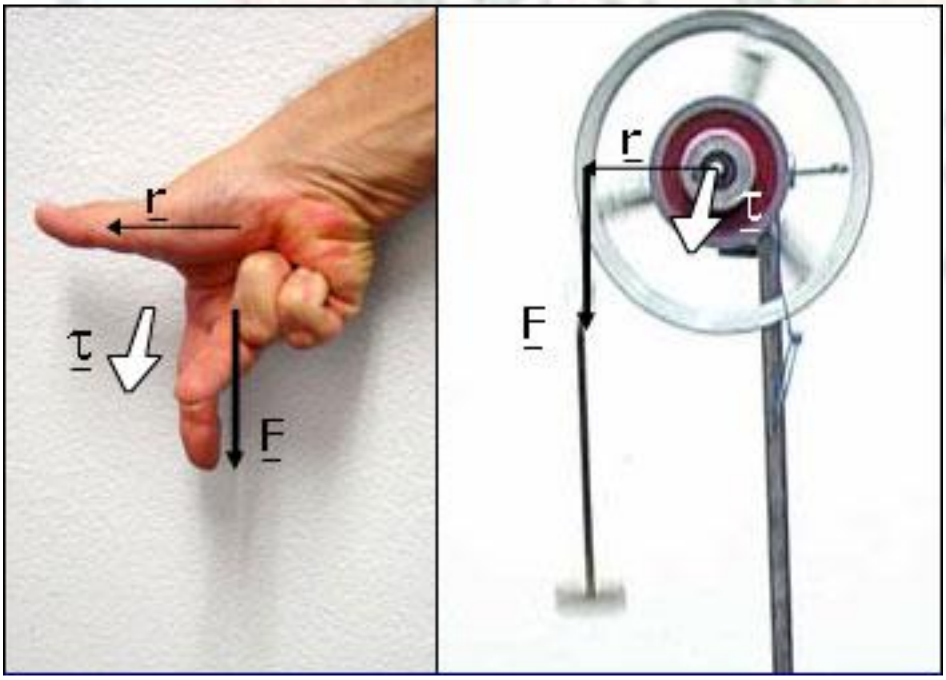
The moment of inertia has S.I units kg. m^2. Using this relation, Eq.(2.11) can now be written as

$$\sum \vec{T} = I\,\vec{\alpha} \qquad (2.14)$$

The angular acceleration of an extended rigid object is proportional to the net torque acting on it. The constant of proportionality is the moment of inertia of the object.

Note that Eq.(2.3) is, in fact, the rotational counterpart to Newton's second law of translational motion represented by $\sum \vec{F} = m\vec{a}$: **the force and mass in linear motion respectively correspond to torque and moment of inertia in rotational motion**. An important difference between m and I is that whereas m depends only on the quantity of matter in an object, I depends on both the quantity of matter and the distribution (through the term r^2) in the rigid body.

The image and the link below show the direction of radius vector, force and torque and the effect of radius vector on the force and torque.



Rotational Dynamics / Mechanics-II

http://www.animations.physics.unsw.edu.au/jw/rotation.htm

Credits: Authored and Presented by Joe Wolfe

Multimedia Design by George Hatsidimitris

Laboratories in Waves and Sound by John Smith

Question

Question Number	Type of question
1	Objective

A net torque is applied to an object. Which one of the following will not be constant?

(a)angular acceleration of the object

(b) moment of inertia of the body

(c) centre of gravity

(d) angular velocity of the object

Correct Answer / Option(s)	
	a) False b) False c) False d) True

Justification/ Feedback for the correct answer

A net torque applied to an object will not change angular acceleration, nor does it change the center of gravity nor the moment of inertia. It only changes the angular velocity. So the correct choice is (d).

Summary

In this lesson you have studied

- the requirements to be imposed on an object under the influence of a number of forces when in mechanical (rotational) equilibrium
- the definition of centre of gravity of a body and locating its coordinates using the principle of moments
- to establish the expression for the torque acting on a particle of mass m revolving in an orbit of radius r in terms of its angular acceleration
- to extend this study for a solid disc(rigid body) rotating about its axis and obtain a relation between torque, angular acceleration and moment of inertia of the disc about the axis of rotation.

Exercises

1. A gun of mass M is initially at rest on a horizontal frictionless surface. It fires a bullet of mass m with velocity v. From the statements given below, tick which one is false and which one is true, giving proper reason.

(a) After firing, the centre of mass of the gun-bullet system moves with a velocity m v /M opposite to the direction of motion.

(b) After firing, centre of mass of the gun-bullet system remains at rest.

Answer: Statement (a) is false but (b) is true. Since there is no external force acting on the gun-bullet system, the centre of mass of the system remains at rest.

2. A system of particles of masses, $m_1, m_2, m_3 \ldots\ldots\ldots\ldots m_n$ are located at the points $x_1, x_2, x_3, \ldots\ldots x_n$ along the x-axis. Which principle do you use to locate the centre of gravity of the system? Write the expression for its centre of gravity.

Answer : We apply the principle of moments, or equivalently equate the torque exerted by weight of the system at the centre of gravity to the sum of the torques acting on the individual particles.The centre of gravity of the system of particles is $x_{CG} = \dfrac{\sum_{i=1}^{n} m_i\, x_i}{\sum_{i=1}^{n} m_i}$

3. Two particles of equal mass move with velocities $\vec{v}_1 = \alpha\, \hat{i}$ *and* $\vec{v}_2 = \alpha\, \hat{j}$. The acceleration of the first particle is $\vec{a}_1 = \beta\, (\hat{i} + \hat{j})$, where α and β are constants. If the acceleration of the second particle is zero, how will the centre of mass of the two particles move?

Answer: $\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{\vec{v}_1 + \vec{v}_2}{2} = \frac{\alpha}{2}(\hat{i} + \hat{j})$, since the two particles have equal mass. Similarly, the acceleration of the centre of mass a $\vec{a}_{CM} = \frac{\beta}{2}\vec{a}_1 = \frac{\beta}{2}(\hat{i} + \hat{j})$.

Since velocity and acceleration vectors are parallel to each other, the centre of mass will move along a straight line.

4. The mass per unit length of a non-uniform rod AB of length varies as m=k x/ L, where k is a constant and x is the distance of any point of the rod from the end A. Which one of the following is a correct statement?

(a) The distance of the centre of mass of the rod from the end A is L/3.

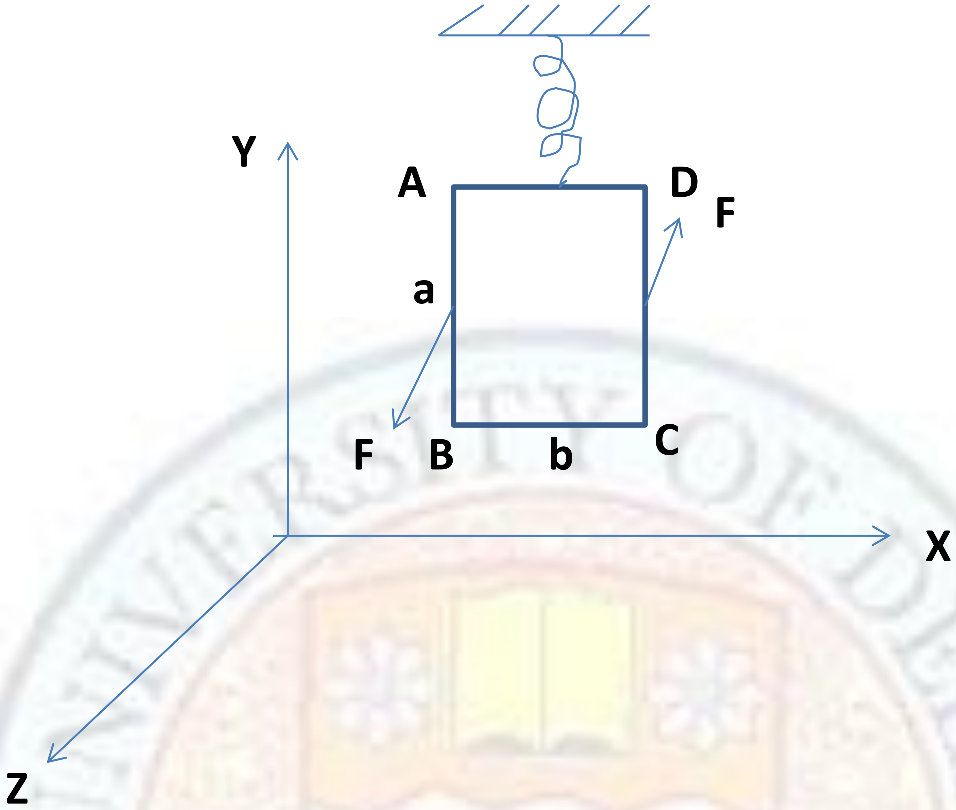
(b) The distance of the centre of mass of the rod from the end A is 2 L/ 3.

Answer: In this case the mass is continuously varying. Therefore, the summation has to be replaced by the integration. Thus

$$x_{CM} = \frac{\int (dM)\, x}{\int (dM)} = \frac{\frac{k}{L}\int_0^L x^2 dx}{\frac{k}{L}\int_0^L x\, dx} = \frac{2L}{3}$$

The correct statement is (b).

5. A rectangular coil, ABCD, of length a and breadth b is suspended in the x-y plane. A force **F** acts along the positive z-axis perpendicular to the length AB and another force of the same magnitude but along the negative z-axis acts normal to the length CD of the coil as shown in the figure. How will the pair of these forces affect the position of the coil?



Answer: Since the two forces have equal magnitude but act in opposite direction, they would form a couple and produce a torque of magnitude F times b causing the coil to rotate (anti-clockwise) about the x-axis.

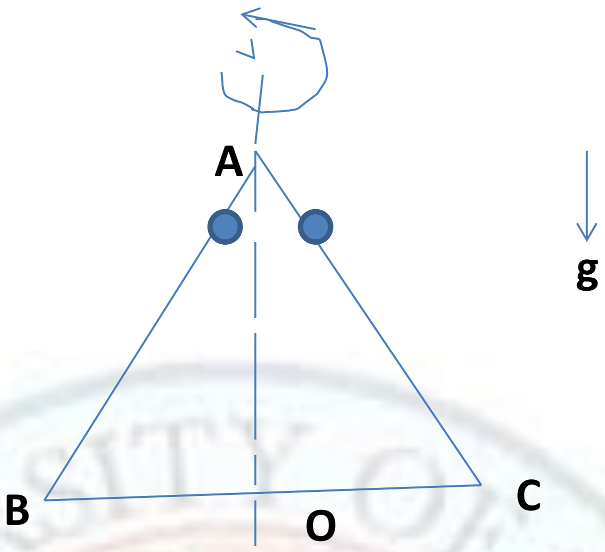
6. A solid sphere is rotating in free space. If the radius of the sphere is increased. Keeping its mass the same, which one of the following will not change?

(a) Moment of inertia (b) Angular momentum

(c) Angular velocity (d) Rotational kinetic energy

Answer: Since no torque acts on the sphere, its angular momentum L=I ω is conserved. If the radius of the sphere is changed, I and hence ω will both change. Also rotational kinetic energy= $\frac{1}{2}I\omega^2$ will also change. So the angular momentum will not change.

7. An equilateral triangle , ABC , formed from a uniform wire has two small identical beads initially located at A. The triangle is set rotating about the vertical axis AO. Then the beads are released from rest simultaneously and allowed to slide down, one along AB and the other along AC as shown in the figure. Neglecting frictional effects, the quantities which are conserved are:



(a) angular velocity and total energy (kinetic plus potential)
(b) total angular momentum and total energy
(c) angular velocity and moment of inertia about the axis of rotation
(d) total angular momentum and moment of inertia about the axis of rotation

Answer: As no external torque acts on the system, the angular momentum L is conserved. As the beads slide down, the moment of inertia of the system will change. From the relation L=I ω, angular velocity ω will change. Since the total energy can not change, the correct choice is (b).

8. A 0.1 kg stone is revolved at the end of a 0.5 m long string at the rate of 2 revolutions per second. If after 25 s, it is making only one revolution per second, find the mean torque.

Answer: The angular momentum L= $mr^2\omega = 0.1\times(0.5)^2\times(2\pi\times 2)$

Torque, T= d L /d t= $mr^2\frac{d\omega}{dt} = 0.1\times(0.5)^2\times\frac{2\pi}{25} = 2\pi\times 10^{-3}$.

9. If A denotes the areal velocity of a planet of mass M, assuming that it has a circular orbit of radius R, estimate its angular momentum.

Answer: Areal velocity, A, is defined as the area swept by the radius vector per unit time. Thus

$$A = \pi R^2 / T, \quad where \quad T = 2\pi/\omega$$
$$\Rightarrow R^2\omega/2, \quad which\ gives\ \omega = 2A/R^2$$

Now , Angular momentum L=I ω, where I is the moment of inertia A= MR^2. Thus, Angular momentum = 2 M A.

10. A molecule consists of two atoms, each of mass m, separated by a distance a. The rotational kinetic energy of the molecule is K and its angular frequency is ω. If I is the moment of inertia of the molecule about its centre of mass, which of the following statements are correct?

(a) $I = ma^2$ (b) $\omega = \frac{1}{a}\sqrt{\frac{K}{m}}$ (c) $I = ma^2/2$ (d) $\omega = \frac{2}{a}\sqrt{\frac{K}{m}}$

Answer : The centre of mass is at a distance of a/2 from each atom, as the two atoms have the same mass. Therefore, the moment of inertia I=2m $(a/2)^2 = ma^2/2$ Also kinetic energy K= $\frac{1}{2}I\omega^2$, which gives $\omega = \frac{2}{a}\sqrt{\frac{K}{m}}$. So the statements (c) and (d) are correct.