Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Rotational Dynamics / Mechanics-IV
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4.1 Introduction / Objectives

We have seen in the preceding lesson that moment of inertia of a rigid body rotating about a given axis plays the same role as mass has in translational motion. An important difference, however, is that moment of inertia of a body depends not only on the quantity of matter but also on the way matter is distributed about the axis of its rotation. In this lesson, we shall study how moment of inertia of certain symmetrical objects can be analytically determined.

Objectives

After studying this lesson you should be able to

- explain the general principle used to determine analytically the moment of inertia of symmetrical objects
- derive the expression to find out the moment of inertia of a thin uniform rod about the axis passing through its centre of mass and perpendicular to the length
- state and prove the theorems of parallel and perpendicular axes
- deduce the expression for the moment of inertia of a thin rectangular lamina about the axis passing through the centre of mass and perpendicular to its plane. [Here you will also be able to learn the use of the theorem of perpendicular axes.]
- use this general procedure to determine the moments of inertia of other symmetrical objects such as (i) a circular ring, (ii) circular disc, (iii) a cylinder, (iv) a sphere .

4.2 Calculation of Moment of Inertia of Some Symmetrical Objects

We have seen that the moment of inertia of a body about an axis is given by (cf., Eq.(2.12, 2.13))

$$I = \sum m r^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...$$

Clearly, an object is supposed to consist of an infinitely large number of point masses at different locations from the axis of rotation. In order to determine the moment of inertia, it is practically not possible to sum the infinite number of terms. However, if we are considering an object of **continuous, homogeneous structure**, it would then be justified to replace the summation by integration. Therefore, exploiting the symmetry of the given object, we can choose an infinitesimally small element for the moment of inertia over which the integration can be performed. We shall here consider a few examples to illustrate this basic approach.

4.2.1 Moment of Inertia of a Thin Uniform Rod

Consider a thin uniform rod, PQ, of length L and mass M and AB the axis of rotation passing through the centre of mass O of the rod and perpendicular to its length, as is shown in the Figure (4.1). Let us suppose a small element of thickness dx at a distance x from O. The mass of this element is (m/L)dx and its moment of inertia about the axis AB passing through O is

$$dI = \frac{M}{L} x^2 dx$$

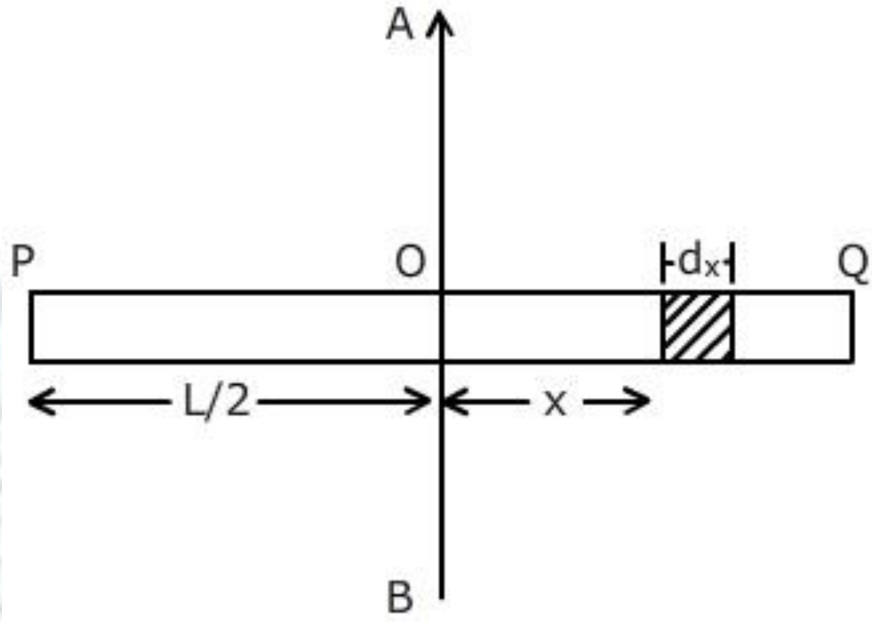


Figure 4.1

The moment of Inertia I of the whole rod about the axis AB will be the sum of the moments of inertia of all such elements lying between x=-L/2 (at the end P) to x=L/2 (at the end Q). Thus

$$I = \int_{-L/2}^{L/2} \frac{M}{L} x^2 dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \frac{M}{L}\frac{2}{3}\left[\frac{L}{2}\right]^3 = \frac{ML^2}{12} \qquad (4.1)$$

We have thus found the moment of inertia of a thin uniform rod about the axis passing through the centre of mass and perpendicular to its length.

Suppose we are required to determine the moment of inertia about one of its ends and perpendicular to the length. One way out is to carry out the above integration from 0 to L, which gives us the result

$$I = \frac{ML^2}{3} \qquad (4.2)$$

Alternatively, this result can also be obtained by using a **theorem of parallel axes**. This is a general theorem, which would be useful in many applications. Let us first discuss this theorem.

4.2.2 Theorem of Parallel Axes

The theorem states that the moment of inertia of a body about any axis is equal to the sum of the moment of inertia about a parallel axis passing through its centre of mass and the product of the mass of the body and the square of the distance between the two axes.

Proof

Suppose we are required to find the moment of inertia about an axis AB which lies in the plane of the paper and this axis is at a distance 'a' from the parallel axis CD passing through the centre of mass O of the body (See Fig. 4.2).

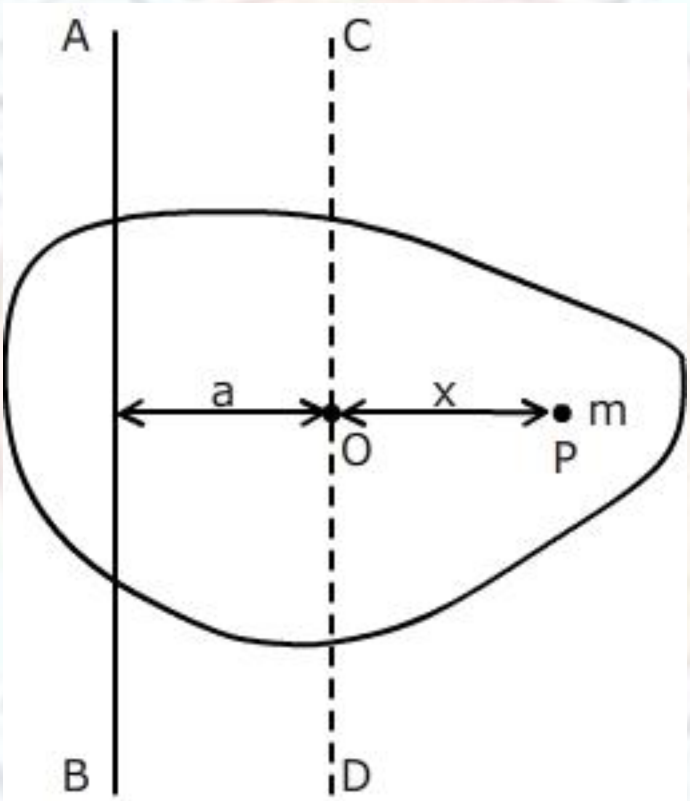


Fig. 4.2

Consider a small element of mass m of the body at the point P, distance x from CD.

The moment of inertia of m about AB

$$I_m = m(x+a)^2$$

Therefore, the moment of inertia of the whole body about AB is

$$I_{AB} = \sum m(x+a)^2 = \sum mx^2 + \sum ma^2 + 2\sum m\,x\,a \quad (4.3)$$

The first term on the right hand side of this equation represents the moment of inertia of the body about the axis passing through its centre of mass,i.e.,

$$I_{CM} = \sum mx^2. \quad (4.4)$$

And, since the distance between the two axes is constant, the second term reduces to

$$\sum ma^2 = a^2 \sum m = Ma^2 \quad (4.5)$$

The third term, $\sum mx$, in fact, represents the sum of moments of all the particle about the axis CD, which passes through its centre of mass. We know that the algebraic sum of all the moments passing through its centre of mass is zero. This proves the theorem stated above:

$$I_{AB} = I_{CM} + Ma^2 \quad (4.6)$$

Exercise

Use Eq.(4.6) to convince yourself that the moment of inertia of a thin uniform rod (considered in the preceding section) about an axis passing through one end and perpendicular to its length is given by Eq.(4.2).

4.2.3 Theorem of Perpendicular Axes

Like the theorem of parallel axes, this theorem is also equally important and useful in determining the moment of inertia about the axes perpendicular to the plane of the objects, given its moments of inertia in the plane.

This theorem states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of moments of inertia of the lamina about the two axes at right angles to each other in its own plane, intersecting each other at the point where the perpendicular axis passes through it.

Let I_x and I_y be the moments of inertia of a plane lamina about OX and OY axes, which lie in the plane of the lamina, then the moment of inertia about the axis which is perpendicular to the plane of the lamina and passing through O is given by

$$I = I_x + I_y$$

Proof

Consider a plane lamina as shown in the figure (Fig.4.3).

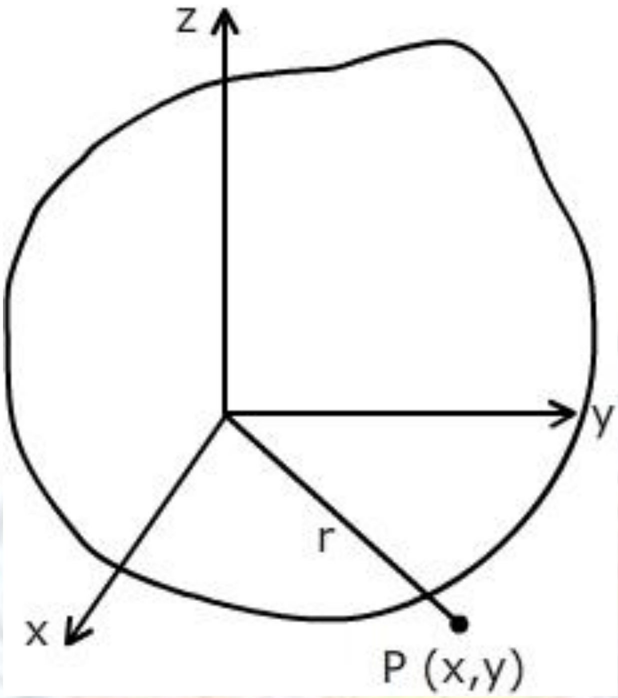


Fig. 4.3

Suppose there is a particle P of mass m lying in the plane of the lamina. Let the particle P be at a distance x from the axis OX and distance y the OY axis. The moments of inertia of the planar body about the x- and y- axis are given by

$$I_x = \sum mx^2, \quad \text{and} \quad I_y = \sum my^2$$

Their sum is

$$I_x + I_y = \sum m(x^2 + y^2) = \sum mr^2 \qquad (4.7)$$

Here r is the distance from the origin, which is also the distance from the z-axis, since the body is in the x-y plane. Therefore, for a body in the x-y plane, we have

$$I_x + I_y = I_z \qquad (4.8)$$

This is the **theorem of perpendicular axes**.

4.2.4 Moment of Inertia of a Thin Rectangular Lamina

(a) About an axis perpendicular to the plane and passing through its centre of mass

Consider a rectangular lamina ABCD of mass M, length l and breadth b placed such that its centre of mass coincides with the origin O, as is shown in the figure (4.4). Let YY’ be the axis parallel to the side AD and passing through the centre of mass about which the moment of inertia is to be determined. Consider a strip of width dx and area b dx at a distance x from the origin. Let its mass be μ b dx, where μ represents the mass per unit area of the lamina.

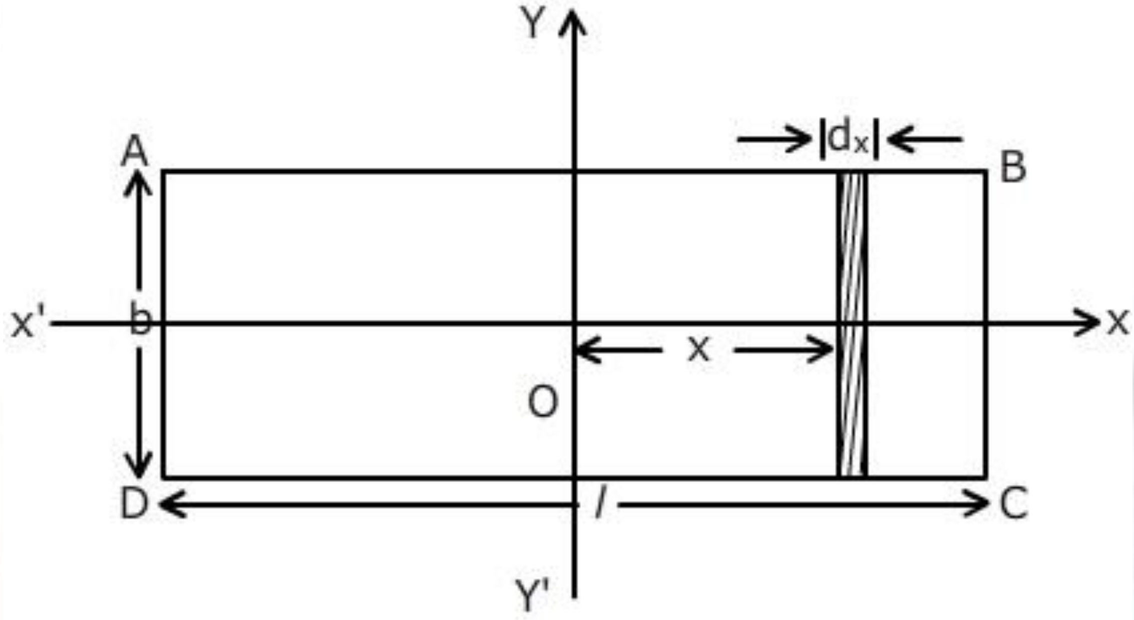


Figure 4.4

The moment of inertia of the strip about YY' is given by

$$dI_Y = \mu b x^2 dx$$

To calculate the moment of inertia of the whole lamina about YY', we integrate the above expression between the limits x=-l/2 and x=+l/2. Thus

$$I_Y = \int_{-l/2}^{l/2} \mu b x^2 dx = \mu b \left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \mu b \frac{2}{3}\frac{l^3}{8} = \frac{\mu b l^3}{12} = \frac{Ml^2}{12}, \qquad (4.9)$$

Since mass of the lamina is M=μ l b.

Exercise

Using this procedure, you can show that the moment of inertia of the lamina about the axis XX' can be written as $I_X = \frac{Mb^2}{12}$ (4.10)

Now you can apply the theorem of perpendicular axes to write that the moment of inertia of a rectangular lamina of mass M , length l and breadth b about the axis perpendicular to the plane of the lamina and passing through its centre of mass can be expressed as

$$I = I_X + I_Y = M\left(\frac{l^2 + b^2}{12}\right) \quad (4.11)$$

4.2.5 Moment of Inertia of a Thin Circular Ring (or a Hoop)

(i) about an axis through its centre and perpendicular to its plane

Let M be the mass and R the radius of the ring or a hoop. Consider a point particle of mass m of the ring (See Fig.4.5). Its moment of inertia about an axis

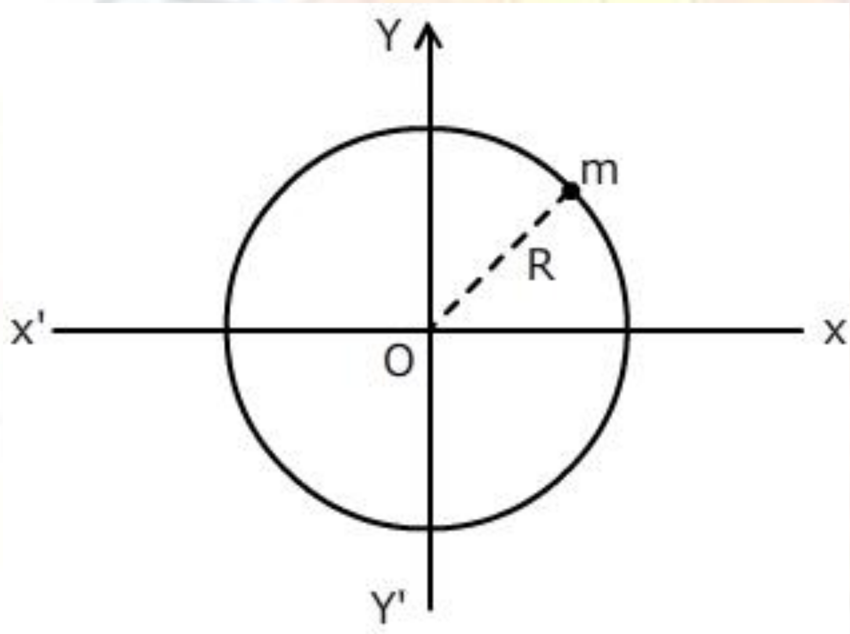


Fig.4.5

passing through the centre O and perpendicular to its plane is mR^2. Therefore the moment of inertia of the entire ring about the given axis will be

$$I = \sum mR^2 = R^2 \sum m = MR^2 \quad (4.12)$$

(ii) about it diameter

The moment of inertia of the hoop is obviously the same about any diameter. If the moment of inertia about the diameter XX' is I, it will also be the moment inertia about the axis YY'. By the application of the theorem of perpendicular axes

$$I = I_X + I_Y = M\,R^2$$

Since $I_X = I_Y \equiv I_D$, where I_D represents the moment of inertia of the hoop about the diameter, we get

$$I_D = M R^2 / 2 \tag{4.13}$$

4.2.6 Moment of Inertia of a Circular Disc

(i) *about an axis passing through its centre and perpendicular to its plane*.

Let us consider a circular disc of radius R and of mass M having mass per unit area $\mu = M / \pi R^2$. Imagine a concentric ring of radius x and infinitesimal thickness dx as shown in Fig.4.6.

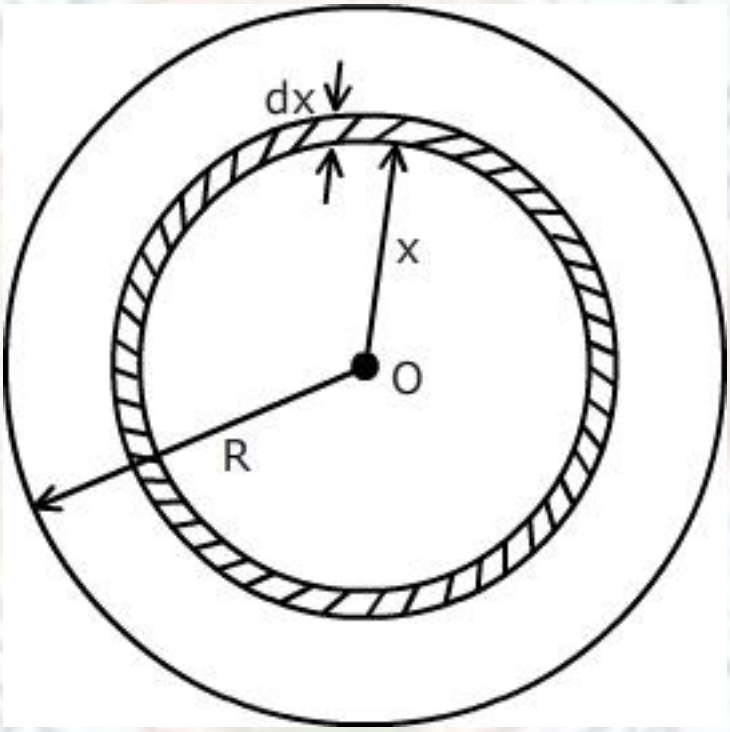


Fig.4.6

Then the area of the ring,

$$dA = 2\pi x\, dx$$

And mass of the ring $= \dfrac{M}{\pi R^2} 2\pi x\, dx = \dfrac{2Mx\, dx}{R^2}$.

Therefore, moment of inertia of this ring about an axis passing through O and perpendicular to the plane of the disc is

$$dI = \frac{2Mx\,dx}{R^2} x^2$$

The moment of inertia of the whole disc about the axis passing through its centre and perpendicular to its plane is obtained by integrating the above from 0 to R,i.e.,

$$I = \frac{2M}{R^2} \int_0^R x^3 dx = \frac{2M}{R^2} \left[\frac{x^4}{4} \right]_0^R = \frac{MR^2}{2} \qquad (4.14)$$

(ii) About a diameter

Since moment of inertia of the circular disc about any diameter is the same, we choose the diameters along x-axis and y-axis and then apply the theorem of perpendicular axes. Thus

$$I_Z = I_X + I_Y = \frac{MR^2}{2}.$$

Since $I_X = I_Y = I_D$, we get, from the above equation,

$$I_D = \frac{MR^2}{4}. \qquad (4.15)$$

Exercise

Show that the moment of inertia of an annular disc of inner radius R_1 and outer radius R_2 about an axis passing through its centre and perpendicular to the plane is

$$\frac{M}{2}(R_1^2 + R_2^2)$$

Hint: Use Eq.(4.49) for the moment of inertia of the circular disc. The only difference in the present case is that now the integration limits will be from R_1 to R_2.

4.2.7 Moment of Inertia of a Solid Cylinder

(i) About its axis

A solid cylinder can be considered as composed of a large number of circular discs placed one above the other. If each disc is of mass m and having the same radius as that of the cylinder, then we have seen above that moment of inertia of each disc about the axis of the cylinder is $\frac{mR^2}{2}$. Thus the moment of inertia of the cylinder about its own axis is given by

$$I = \sum \frac{mR^2}{2} = \frac{R^2}{2} \sum m = \frac{MR^2}{2} \qquad (4.16)$$

(ii) About an axis perpendicular to the symmetrical axis and passing through its centre of mass

Let us consider the cylinder of length l with its symmetrical axis along the XX', placed such that its centre of mass coincides with the origin O, as is shown in the Fig.(4.7) and let YY' be the axis of rotation perpendicular to the symmetrical axis and passing through the centre.

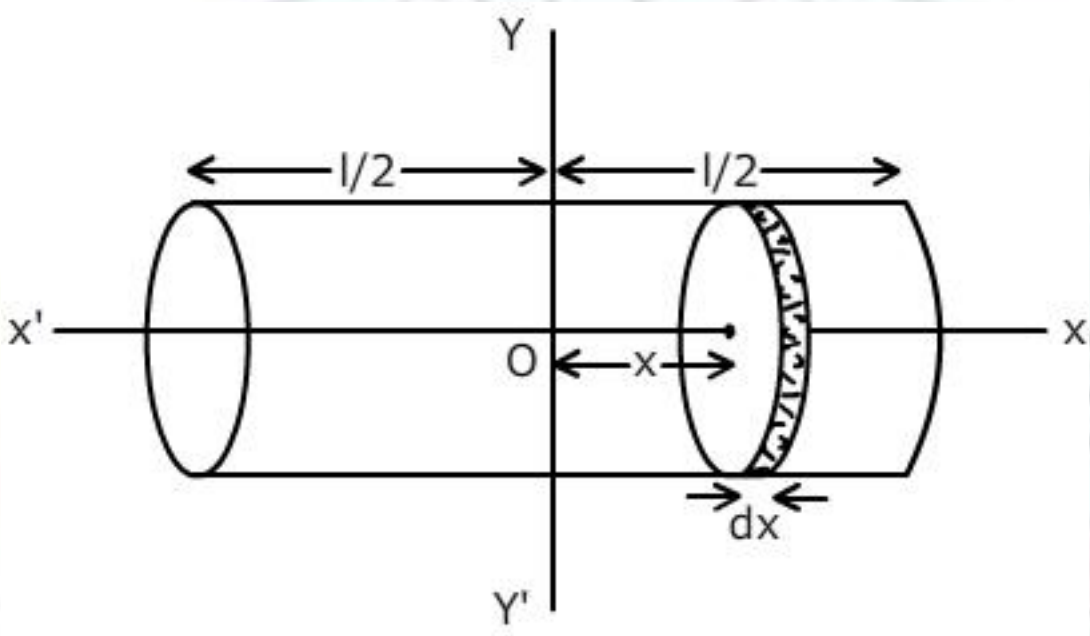


Fig.4.7

Let R be the radius of the cylinder having mass M. Imagine a thin disc of thickness dx at a distance x from O. Moment of inertia of this about its diameter is

$$= \frac{M}{l} dx \frac{R^2}{4}$$

(cf., Eq(4.14)). The moment of inertia of this disc about the YY' axis can be obtained by applying the theorem of parallel axis. Thus

$$dI = \frac{M}{l} dx \frac{R^2}{4} + \frac{M}{l} dx\, x^2$$

Now we can get the moment of inertia of the cylinder about YY' axis by integrating the above expression over the limits from –l/2 to +l/2, which gives

$$I = \frac{M}{l}\int_{-l/2}^{l/2}\left[\frac{R^2}{4} + x^2\right]dx = \frac{M}{l}\left[\frac{R^2 x}{4} + \frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{M}{l}\left[\frac{R^2 l}{4} + \frac{l^3}{12}\right]$$

$$= M\left(\frac{R^2}{4} + \frac{l^2}{12}\right) \qquad (4.17)$$

4.2.8 Moment of Inertia of a Sphere

(i) about a diameter

Let us represent the section of a sphere of mass M and radius R with centre O as shown in the Fig.(4.8). Let XX' be the diameter about which the moment of inertia is to be determined. Consider a very thin circular disc of thickness dx at a distance x from the centre as shown in the figure. If y represents its radius and ρ the density, then its mass is given by

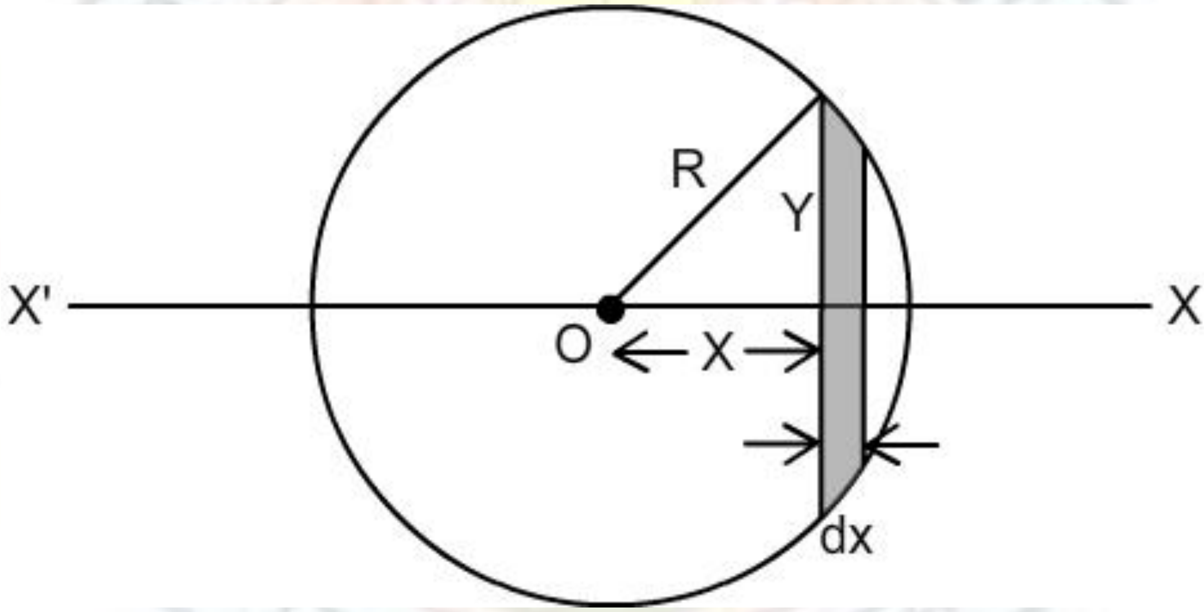


Fig.4.8

$$d M = \pi y^2 dx \rho \qquad (4.18)$$

Therefore, the moment of inertia of the disc about XX' is

$$d I = \frac{d M y^2}{2} = \frac{\pi y^2 dx \rho y^2}{2} \qquad (4.19)$$

Since the radius of this disc is $y = \sqrt{(R^2 - x^2)}$, we get

$$d\,I = \frac{\rho\,\pi(R^2 - x^2)^2\,dx}{2} \qquad (4.20)$$

Hence the moment of inertia of the sphere about XX' can be obtained by integrating the above expression between the limits x=-R to x=R. Thus

$$I = \int_{-R}^{R} \frac{\pi\rho}{2}(R^2 - x^2)^2\,dx = \frac{\pi\rho}{2}\int_{-R}^{R}(R^4 + x^4 - 2R^2x^2)\,dx$$

On integration we get

$$I = \frac{\pi\rho}{2}[R^4x - 2R^2\frac{x^3}{3} + \frac{x^5}{5}]_{-R}^{R} = \frac{8\pi\rho R^5}{15}$$

Since $\rho = \dfrac{M}{4\pi R^3/3}$, the final result is $I = \dfrac{2}{5}M\,R^2$ (4.19)

Value Addition: Moment of Inertia of various objects about their axis of rotation can be visualised through these animations shown in the website link given below:

http://www.animations.physics.unsw.edu.au/jw/rotation.htm#I

Credits: Authored and Presented by Joe Wolfe

Multimedia Design by George Hatsidimitris

Laboratories in Waves and Sound by John Smith

Exercise

Use the theorem of parallel axis to find the moment of inertia of a sphere about a tangent to the sphere.

Summary

In this lesson you study

- a general procedure which is employed to determine the moment of inertia of some symmetrical objects
- to state and prove the theorems of parallel and perpendicular axes
- to obtain the expressions for the determination of moments of inertia of certain typical symmetrical objects like a rectangular lamina, a circular disc, a cylinder, a sphere and a spherical shell etc.,
- to learn the use of the theorems of parallel and perpendicular axes

Rotational Dynamics / Mechanics-IV

Exercises:

1. Indicate which of the following statements are true / false, justifying your answer in each case.

(a) For a particle of a rotating rigid body v=r ω. It implies (i) $\omega \propto (1/r)$, $(ii) v \propto r$.

(b) When angular momentum of a system is conserved, it follows rotational kinetic energy is automatically conserved.

(c) Value of radius of gyration of a body is independent of the axis of rotation.

Answer: (a) As ω= $2\pi/T$= constant, it does not depend on r, so (i) is false. However, (ii) is true.

(b) Rotational kinetic energy $KE = \frac{1}{2} I\omega^2 = \frac{L^2}{2I}$. When L is conserved, KE is conserved only if I remains constant. The statement is false.

(c) Radius of gyration is the root mean square distance of particles of the body from the axis of rotation. So statement (c) is false.

2. Indicate which of the following statements are true / false, giving reason in each case.

(a) The moment of inertia of a rigid body reduces to its minimum value, when the axis of rotation passes through its centre of gravity.

(b) Moment of inertia of a circular ring about a given axis is more than moment of inertia of the circular disc of same mass and same size about the same axis.

(c) In the formation of a neutron star, spin angular velocity increases because of conservation of rotational kinetic energy.

Answer: (a) This is true, because the weight of a rigid body always acts through its centre of gravity. This follows from the theorem of parallel axes.

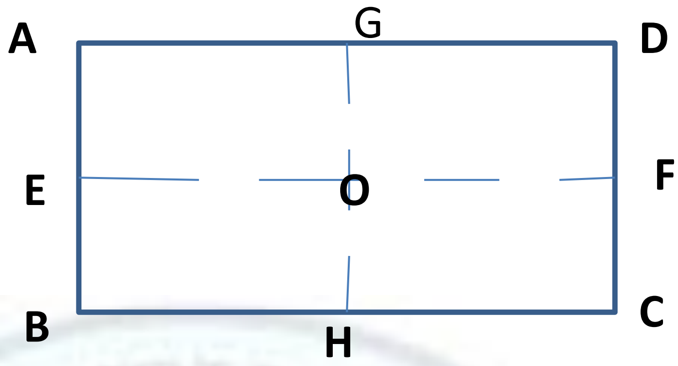
(b) This is true because in the case of circular ring, the mass is concentrated on the rim – at maximum distance from the axis.

(c) The statement is true but the reasoning is false. In the formation of neutron star, a heavy contraction occurs on account of gravitational pull. Moment of inertia decreases. As L=Iω, since L is conserved, ω increases.

3. A solid sphere and a solid cylinder have the same mass M and the same radius R. If torques of equal magnitudes are applied to them for the same time, which would acquire greater angular speed? Explain.

Answer: For cylinder, $I_C = \frac{1}{2} M R^2$ and for sphere $I_S = \frac{2}{5} M R^2$. And the torque T=I α or α=T/I i.e., for a given T, α is inversely proportional to I. Since $I_C > I_S$, therefore $\alpha_S > \alpha_C$. As the torque is applied on the two for the same time, the sphere would acquire greater angular speed than cylinder.

4. In the rectangular lamina ABCD, the side AB is a and side BC is 2 a. The moment of inertia of the lamina is minimum along the axis passing through



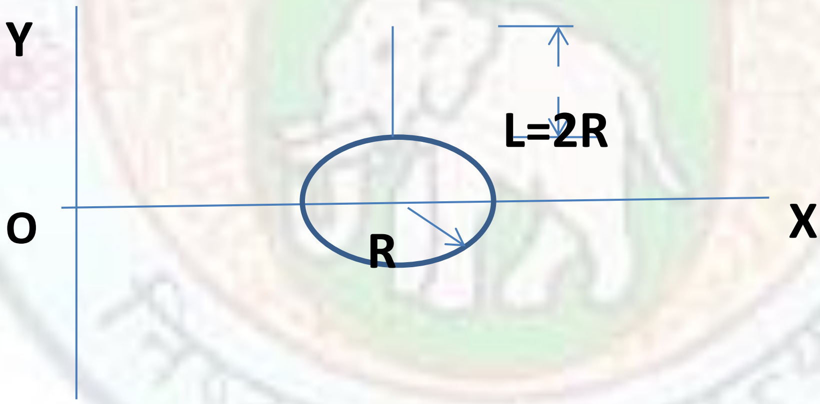
(a) BC (b) AB (c) FE (d) GH

Ansswer

$$I_{BC} = \frac{m(AB)^2}{3} = \frac{ma^2}{3};\ I_{AB} = \frac{m(BC)^2}{3} = \frac{4a^2}{3};\ I_{FE} = \frac{m(AB)^2}{12} = \frac{m(a)^2}{12};\ I_{GH} = \frac{m(BC)^2}{12} = \frac{ma^2}{3}$$

Thus the moment of inertia about FE is minimum. Correct choice is (c).

5. A rigid body consists of a circular hoop of mass M and radius R and a thin long uniform rod of length 2R and mass M. Find out the moment of inertia of the system about the x-axis.



Answer: Moment of inertia of the about x-axis= $\frac{1}{2}MR^2$

Moment of inertia of the rod perpendicular to its length and passing through its centre = $\frac{1}{12}M(2R)^2 = \frac{MR^2}{3}$

Moment of inertia of the rod passing through the centre of the hoop=

$\frac{1}{3}MR^2 + m(2R)^2$

Net moment of inertia of the system= $(\frac{1}{2}+\frac{1}{3}+4)MR^2 = \frac{29}{6}MR^2 = 4.83MR^2$.

6. A uniform bar of length 6 a and mass 8 m lies on a horizontal frictionless table. Two point masses m and 2 m moving in opposite directions but in the same horizontal plane with speeds v and 2 v respectively strike the bar at distance a and 2 a from one end and stick to the bar after collision. Which of the following statements are true?

(a) The velocity of the centre of mass is zero.

(b) The angular speed of the bar with the masses stuck to it is v/(5 a).

(c) The moment of inertia of the bar with masses stuck to it about the axis passing through the end of the bar and perpendicular to its plane is 30 m$(a)^2$

(d) The total energy of the bar is $\frac{3}{5}mv^2$.

Answer: Since no external force is applied, linear momentum is conserved,i.e.,

(8 m +m+2m) $v_{CM}=2m(-v)+m(2v)+8m\times 0$, which gives $v_{CM}=0$.

The moment of inertia of the system is

$$2ma^2+m(2a)^2+\frac{1}{12}(8m)\times(6a)^2=30ma^2$$

Also the angular momentum of the system is conserved as no torque is applied, which gives 2m v×a+m×2v×2a=Iω. This gives ω=v/(5a).

The system has no translational energy, only the kinetic energy of rotation, which is

$\frac{1}{2}I\omega^2=\frac{1}{2}30ma^2\times\left(\frac{v}{5a}\right)^2=\frac{3}{5}mv^2$. All the statements given above are correct.

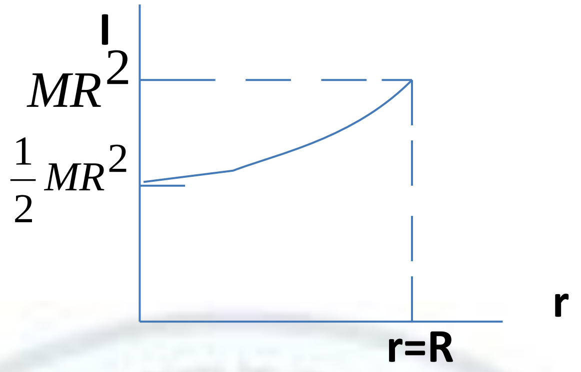
7. A thin uniform circular disc of mass M and radius R is rotating in a horizontal plane about its axis passing through its centre and perpendicular to its plane with angular velocity ω. Another disc of same dimensions but of mass M/4 is placed gently on the first disc coaxially. What is the angular velocity of the system now?

Answer: As no torque is applied $I_1\omega_1=I_2\omega_2$

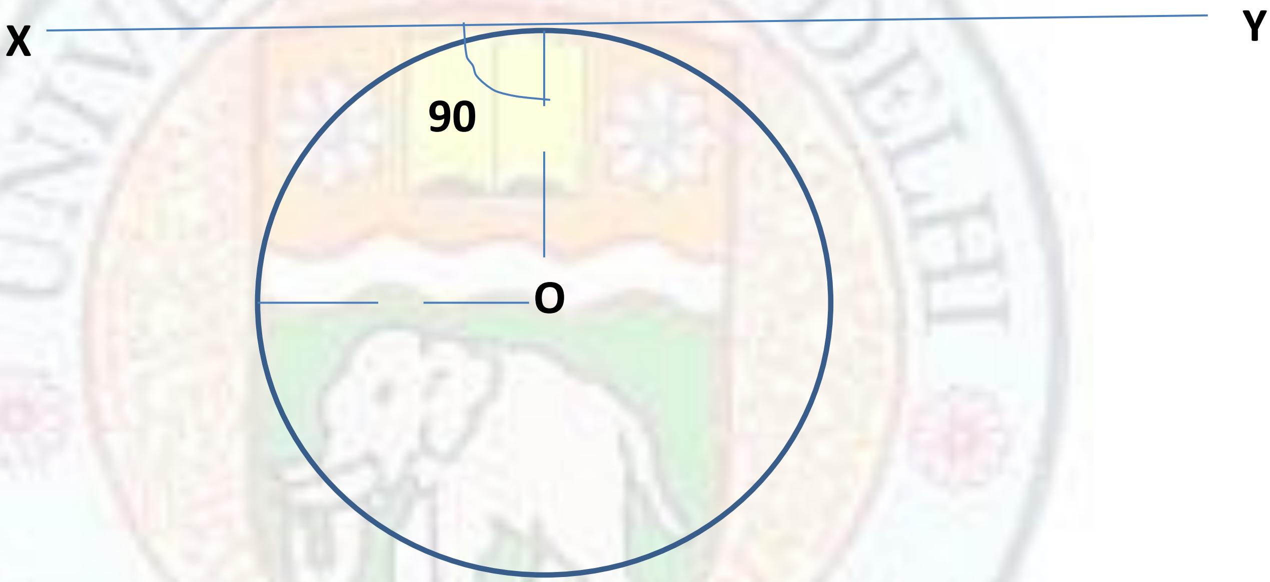
Therefore $\omega_2=\dfrac{I_1\omega_1}{I_2}=\dfrac{\frac{1}{2}MR^2\omega}{(\frac{1}{2}MR^2+\frac{1}{2}\frac{M}{4}R^2)}=4\omega/5$.

8. You are given different annular discs of same mass and outer radius R but different inner radii r. Draw a plot showing the variation of their moment of inertia about an axis passing through their centre of gravity and perpendicular to their plane versus r.

Answer: The moment of inertia of the annular disc $I=\dfrac{M(R^2-r^2)}{2}$. The plot of I versus 'r' is obtained as shown .



9. A thin wire of length L and uniform linear mass density ρ is bent into a circular loop with centre O as shown. Find out the moment of inertia of the loop about the axis XY



Answer: If m is the mass of the loop and r is its radius, then the moment of inertia of the loop about an axis passing through the centre O is

$$I_O = \frac{1}{2}mr^2.$$

Using the theorem of parallel axes, its moment of inertia about XY is

$$I = I_O + mr^2 = \frac{3}{2}mr^2.$$

The mass of the loop is m=ρ L and radius is r=L/2π, therefore

$$I = \frac{3}{2}\rho L \times \left(\frac{L}{2\pi}\right)^2 = \frac{3\rho L^3}{8\pi^2}.$$

10. The angular velocity of a body changes from ω_1 *to* ω_2 without applying any torque but by changing the moment of inertia about its axis of rotation. What would be the ratio of the corresponding radii of gyration?

Answer: If no torque acts, the angular momentum must be conserved, i.e., $I_1\ \omega_1 = I_2\ \omega_2$. If K_1 *and* K_2 are the corresponding radii of gyration, then $I_1 = M\,K_1^2$ *and* $I_2 = MK_2^2$. Thus $M\,K_1^2\,\omega_1 = MK_2^2\omega_2$. This gives

$$\frac{K_1}{K_2} = \frac{\sqrt{\omega_1}}{\sqrt{\omega_2}}$$