Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Rotational Dynamics / Mechanics-V
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5.1 Introduction

This lesson extends the study of rotational motion to obtain the expression for the kinetic energy of a rotating body and considers, as an application, the mechanics of a fly wheel – a device which is commonly used to store rotational energy and has wide applications in many instruments. You will also find how, in the case of irregular rigid bodies, the expression for the moment of inertia can be generalized to what, in mathematical language, is known as a tensor of second rank.

Objectives

After studying this lesson you should be able to

- know how to derive the expression for the kinetic energy of a rigid body rotating with angular velocity ω
- learn the mechanics and use of a flywheel and describe the method of finding its moment of inertia
- show and express the moment of inertia of an irregular body as a tensor of rank two

5.2 Kinetic Energy of a Rotating Body

To obtain the expression for kinetic energy of a rigid body rotating with angular velocity ω about the axis AB, we again focus on a particle P of mass m_1 at a distance r_1 from the axis. The kinetic energy of this particle is

$$= \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 r_1^2 \omega^2 . \qquad (5.1)$$

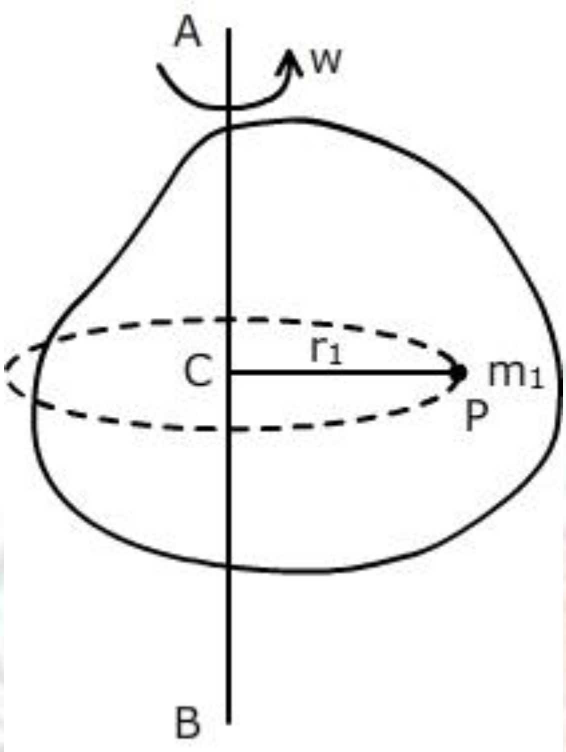


Fig.5.1

The body is made up of a large number of such particles. Therefore, the total kinetic energy of the body

= the sum of the kinetic energy of all the particles composing the body

Since every particle is rotating with the same angular velocity hence the total kinetic energy in mathematical form is given by,

$$\frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + \frac{1}{2}m_3 r_3^2\omega^2 + \ldots\ldots\ldots\ldots = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2 = \frac{1}{2}I\,\omega^2 \qquad (5.2)$$

This gives the expression for the kinetic energy of a rigid body rotating with angular velocity ω in terms of the moment of inertia of the body. Comparing the expressions for the kinetic energy in rotational and translational motion, we again find that the moment of inertia is the rotational analogue to the mass.

In an earlier section, we obtained a relation between angular momentum L of a rigid body rotating with angular velocity ω , viz., (cf.,(3.8)), given by L=I ω. Substituting for ω in Eq.(5.2), we get

$$\text{Kinetic Energy (K.E) of a rotating body} = L^2/2I, \qquad (5.3)$$

relating it with angular momentum about the same axis.

5.2.1Value Addition   Do you know a Flywheel?

A flywheel is essentially a heavy wheel having a large amount of inertia with is mass mostly concentrated on the rim. It is basically a rotating mechanical device used to store rotational energy. Because of having significantly large moment of inertia, they can resist changes in rotational speed. The amount of energy stored is directly proportional to the square of rotational speed. Flywheels have wide applications in stationary engines and various types of instruments of everyday use. The common uses of a flywheel include:

(i) To provide continuous energy when the energy source is discontinuous;
(ii) To deliver energy at rates beyond the ability of a continuous energy source;
(iii) To control the orientation of a mechanical system by transferring the angular momentum of the flywheel

A flywheel is also used in the laboratory to carry out experiment to determine its moment of inertia. It is typically made of steel and is mounted on two ball bearings as shown in the figure (Fig.5.2).

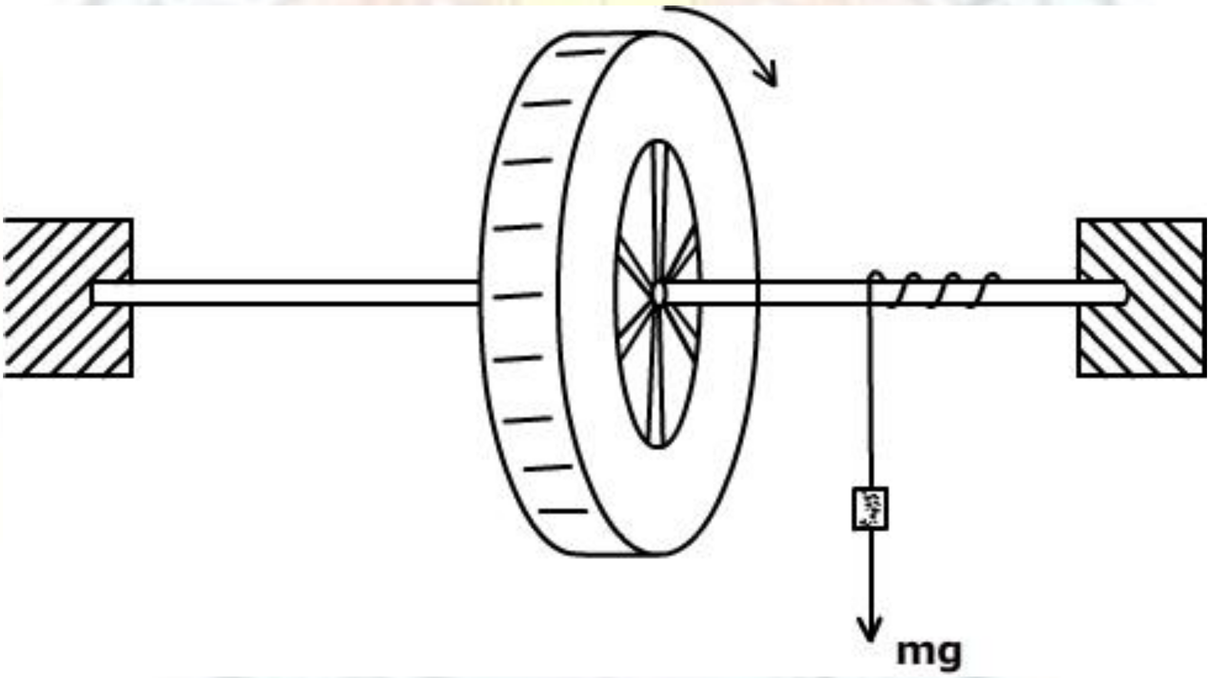
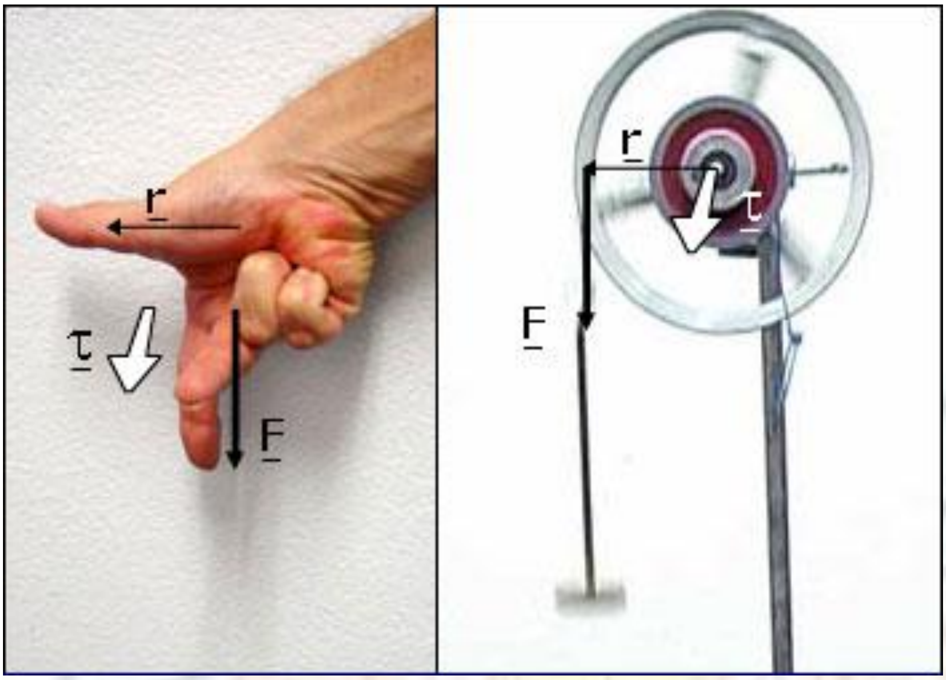


Fig. 5.2

A small loop made on one end of a cord is slipped onto a small peg on the axle around which the whole length of the cord is wound. At the other end a small mass is attached to the cord. The mass is allowed to fall under gravity. As it loses its potential energy, it makes the flywheel to rotate and gain kinetic energy.



Click on the link below to view some interesting animations on Flywheel

http://www.animations.physics.unsw.edu.au/jw/rotation.htm

Credits: Authored and Presented by Joe Wolfe

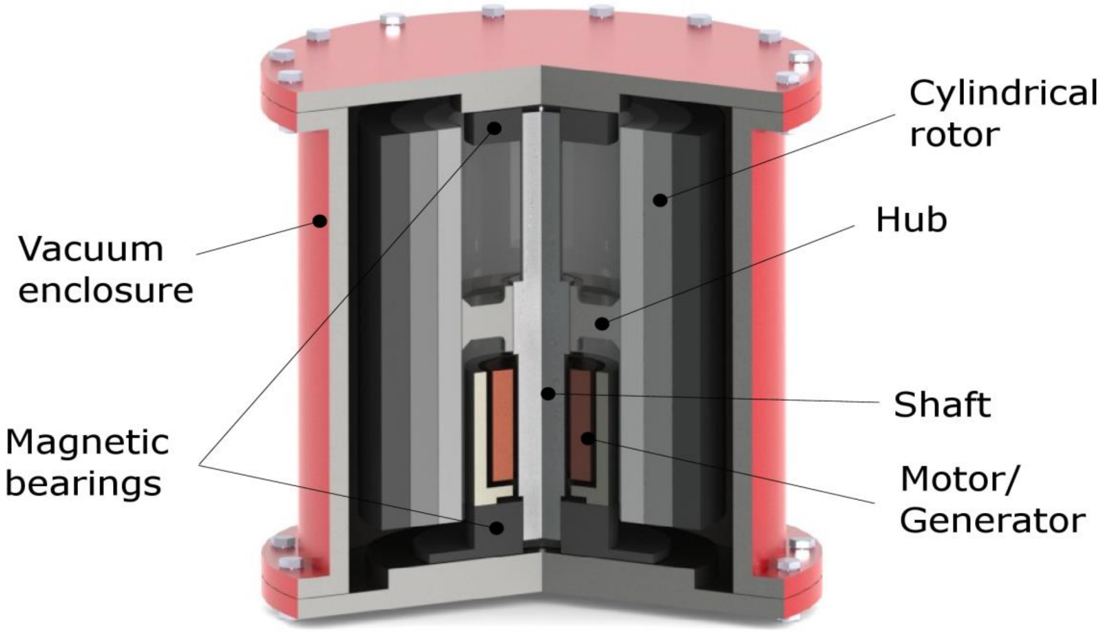
Multimedia Design by George Hatsidimitris

Laboratories in Waves and Sound by John Smith

One of the main applications of the Flywheel is as an energy storage device, which is increasingly be used in industries like Automobile, Aeronautics and Aerospace, Heavy Electrical and Earth moving, etc. The main principle is to use the rotary motion of the equipment being used to accelerate the flywheel (rotor) to a very high speed and then store the energy in the machinery as rotational energy (potential energy). The machinery then extracts this stored energy by converting it into rotational kinetic energy thereby slowing down the flywheel to adhere to law of conservation of energy. This cycle is repeated again and again and in each such cycle the flywheel helps in conserving and reusing the rotational energy. In order to understand this better students are advised to visit the following links:

http://www.youtube.com/watch?v=LC0pHkstuF8

http://en.wikipedia.org/wiki/Flywheel_energy_storage



The procedure for the determination of the moment of inertia of a flywheel in the laboratory can be best described through an example which is given below:

Example

A chord is wound round the horizontal axle of radius 1.5 cm of a flywheel and a mass of 2 kg is attached to the free end of the rod. Starting from rest, the mass is released from the axle after passing through 100 cm. After the mass is released, the flywheel makes 15 turns in 6.0 seconds before coming to rest. Calculate (i) the kinetic energy of the mass at the moment of the release; (ii) the moment of inertia of the flywheel.

Solution

Let I be the moment of Inertia of the flywheel, v be the velocity acquired by the mass falling through height h. Then, according to the law of conservation of energy,

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + NW \quad , \qquad \text{(i)}$$

where ω is the angular velocity of the flywheel, W is the work done per revolution against friction, N is the number of revolutions made by the wheel before the mass gets detached from the axle.

If the wheel makes n rotations before coming to rest after the mass is detached from the axle, work done against friction during the n revolutions, i.e., n w must be equal to the kinetic energy of rotations of the wheel, which has been used up doing this work. Thus

$$nW = \frac{1}{2}I\omega^2, \quad \text{which gives} \quad W = \frac{1}{2}I\omega^2/n.$$

Substituting for W in the expression (i), we have

$$2mgh = mv^2 + I\omega^2 + NI\omega^2/n \quad \text{Or} \quad 2mgh - mv^2 = I\omega^2(1 + N/n)$$

From the above equation, we get

$$I = \frac{2mgh - mv^2}{\omega^2(1+N/n)} = \frac{2mgh - mr^2\omega^2}{\omega^2(1+N/n)}$$

$$= \frac{(2mgh/\omega^2) - mr^2}{(1+N/n)}$$

where v=r ω, and r is the radius of the axle. As the wheel makes n rotations, i.e., it describes an angle of 2π n in time t, its average angular velocity is equal to 2 π n/t. This should be equal to the average angular velocity of the wheel , viz., ω/2. Thus ω=4π n/t. Substituting this for ω in the above equation, we get

$$I = \frac{m\left[2ght^2/(16\pi^2 n^2) - r^2\right]}{(1+N/n)} = m\left(\frac{n}{n+N}\right)\left(\frac{ght^2}{8\pi^2 n^2} - r^2\right)$$

Let us now substitute the values given in the question in this equation to determine the moment of inertia of the flywheel.

m=2.0 kg; h= 1 m; r=1.5$\times 10^{-2} m$; n=15 and t=6 s. To find N, we use

$$N=\frac{length\ of\ the\ cord}{2\pi r}=\frac{1}{2\pi\times 1.5\times 10^{-2}}=\frac{100}{3\times\pi}=10.6$$

$$I=2\times\frac{15}{25.6}\left(\frac{9.8\times 1\times 36}{8\times\pi^2\times 225}-2.25\times 10^{-4}\right)=\frac{15}{12.8}\times 10^{-2}=1.17\times 10^{-2}\,kg.m^2$$

Kinetic Energy of the mass at the time of release from the axle= $\frac{1}{2}m\times v^2=\frac{1}{2}m\times(r\,\omega)^2=\frac{1}{2}m\times r^2\times(\frac{4\pi\, n}{t})^2$

$$=\frac{1}{2}\times 2\times 2.25\times 10^{-4}(\frac{4\times\pi\times 15}{6})^2=2.25\times 10^{-2}\times\pi^2=22.2\times 10^{-2}\,J$$

5.3 Moment of Inertia of an Irregular Body – A Tensor of Rank Two

Suppose we consider a rigid body having an irregular shape like that of a potato. Any such irregular body has three mutually perpendicular axes passing through its centre of mass. Let the moments of inertia of the body along these axes be different. These axes are called the **principal axes** of the body. **They have an important property that if the body is rotating about one of them, its angular momentum is in the same direction as the angular velocity.** For a body having axes of symmetry, the principal axes will be along the symmetry axes.

Let us take the x-, y-, and z-axes along the principal axes and suppose the corresponding principal moments of inertia are represented by I_{xx}, I_{yy}, I_{zz} (you will soon realize why the moment of inertia is designated here by the subscripts xx, etc.). Now, if the body is rotating with angular velocity $\vec{\omega}$, we resolve it into components ω_x, ω_y and ω_z along x, y and z axes as is shown in Fig.(5.3).

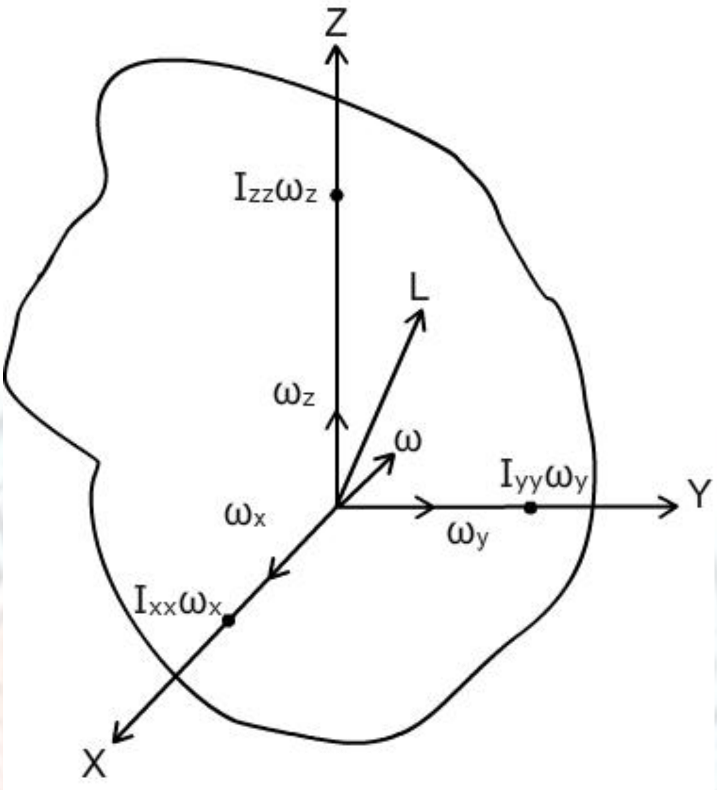


Fig. 5.3 An irregular body rotating about an axis where the angular velocity vector ω is not along the angular velocity vector L.

Since $I_{xx}\,\omega_x\,,\, I_{yy}\,\omega_y \text{ and } I_{zz}\,\omega_z$ now represent the x, y and z components of the angular momentum, we can express the angular momentum of the body as

$$\vec{L} = I_{xx}\,\omega_x\,\hat{i} + I_{yy}\,\omega_y\,\hat{j} + I_{zz}\,\omega_z\,\hat{k} \qquad (5.4)$$

The kinetic energy of rotation is

$$KE = \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right)$$
$$= \frac{1}{2}\vec{L}.\vec{\omega} \qquad (5.5)$$

The situation that we discussed above is yet not a general one. We can have, in principle, an irregular body whose x- component of angular momentum is not only proportional to ω_x but also proportional to the components $\omega_y \text{ and } \omega_z$. If the moments of inertia of the body associated with the components $\omega_y \text{ and } \omega_z$ are represented by $I_{xy} \text{ and } I_{xz}$, then the x- component of angular momentum, L_x, can be written as

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \qquad (5.6)$$

Similar expressions for the components $L_y \text{ and } L_z$ of the y- and z- components of the angular momentum would be

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \qquad (5.7)$$

We have thus seen that the moment of inertia of a body can, in general, have **nine** different components. In mathematical terms, such a quantity is known as a **tensor of second rank in three dimensions.** We can thus conclude from the above study that moment of inertia of an irregular body is, in general, a **second rank tensor** in three dimensions. You will have the occasion to encounter many physical quantities of this nature from different branches of physics, which, from mathematical standpoint, should be regarded not as scalars or vectors but as tensors.

In tensor notation, the kinetic energy of a rotating irregular body can be expressed as

$$KE = \frac{1}{2}\sum_{i\,j} I_{i\,j}\,\omega_i\,\omega_j \quad , \qquad (5.8)$$

where the subscripts (i, j), each of which stand for x, y, z have to be summed over.

Summary

In this lesson, you learn

- to deduce the expression for the kinetic energy of a rotating body
- how rotational kinetic energy is stored in a Flywheel and how its moment of inertia can be determined
- that moment of inertia of an irregular body can, in general, be represented by a tensor of rank two.

Exercise / Questions :

1. Kinetic energy of a rotating body having angular momentum L and momentum of inertia I is given by

(a) L I/2 (b) $\frac{1}{2}LI^2$ (c) L /(2 I) (d) $\frac{L^2}{2I}$

Answer: KE= $\frac{1}{2}I\,\omega^2$; $\quad$ since $\quad L = I\,\omega,\quad$ by substituting, we get $KE = \frac{L^2}{2I}$.

Therefore correct choice is (d).

2. Why does a flywheel have a significantly large moment of inertia?

Answer: In order to resist changes in rotational speed.

3. Under what condition a flywheel does not have angular momentum $\vec{L}$ along the angular velocity vector $\vec{\omega}$?

Answer : When it is fixed to a shaft in a lopsided manner and the plane of the wheel is not perpendicular to the axis of rotation.

4. A flywheel of mass 500 kg and 1 m diameter revolves about its axis. Its frequency of revolution is increased by 18 in 5 s. Find the torque applied.

Answer: The torque T= I (dω /dt). $I = \frac{1}{2}MR^2 = \frac{1}{2}\times(500)\times(0.5)^2 = 62.5\,kg\,m^2$.

Now d ω/dt=2π×18/5=7.2 π. Thus T=62.5×7.2π=$1.4\times10^3 N-m$.

5. Write two common uses of a flywheel.

Answer: (i) to provide continuous energy;

(ii) to control the orientation of a mechanical system by transferring angular momentum of the flywheel.

6. What is the significance of the principal axes of a rigid irregular body?

Answer: The principal axes of a body have a property that if the body is rotating about one of them, its angular momentum is in the same direction as the angular velocity.

7. When an irregular body is rotating about an axis where the angular velocity vector is not along the angular momentum vector, what would be the expression for the angular momentum $\vec{L}$ in terms of the components of the angular velocity vector? Hence write the expression for its kinetic energy.

Answer: We resolve the angular velocity vector $\vec{\omega}$ into the three components ω_x, ω_y, *and* ω_z along the three mutually perpendicular principal axes, say, x-, y- and z- passing through its centre of mass. If I_{xx}, I_{yy} *and* I_{zz} denote the three different moments of inertia along these axes, then the components of the angular momentum, $L_x = I_{xx}\,\omega_x$, $L_y = I_{yy}\,\omega_y$ *and* $L_z = I_{zz}\,\omega_z$ so that the angular momentum vector

$$\vec{L} = I_{xx}\,\omega_x\hat{i} + L_{yy}\,\omega_y\,\hat{j} + L_{zz}\omega_z\hat{k}\,.$$

Now, since $\vec{\omega} = \omega_x\hat{i} + \omega_y\,\hat{j} + \omega_z\hat{k}$ and kinetic energy KE = $\frac{1}{2}\vec{L}.\vec{\omega}$, therefore

$$KE = \frac{1}{2}(I_{xx}\;\omega_x^2 + I_{yy}\;\omega_y^2 + Izz\omega_z^2)\,.$$

8. Why is the moment of inertia of an irregular rigid body regarded as a tensor and not a simple scalar?

Answer: This is because in the case of an irregular body the x-component of the angular momentum of the body may not depend just on the x-component of the angular velocity but could depend on the y- and z- components of the angular velocity also. So for each component of the angular velocity, the moment of inertia about the x-axis may be different for each component of angular momentum vector.