Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Rotational Dynamics / Mechanics-VI
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Rotational Dynamics / Mechanics-VI

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6.1 Introduction

This lesson describes the mechanics of rolling bodies. Specifically, it discusses some basic features involving the motion of spherically symmetric objects (a) rolling on a plane surface, (b) rolling down an inclined plane without slipping with no loss of energy due to friction and (c) rolling down an inclined plane without slipping of a cylindrical object taking the effect of friction into account .

Objectives

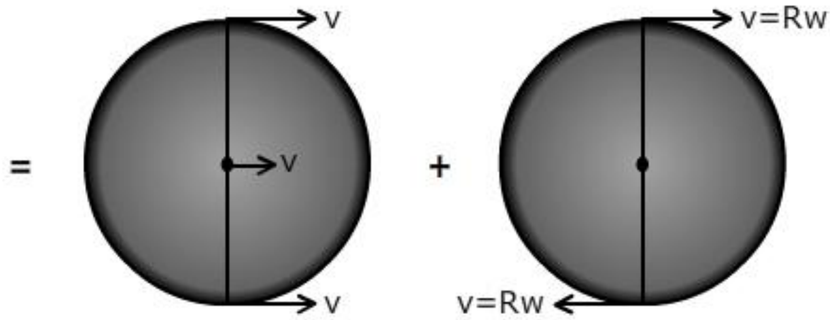
After studying this lesson you should be able to

- learn the motion of bodies having circular symmetry rolling on a plane surface
- show that kinetic energy of rolling consists of two parts: one due to rotational motion and the other of translational motion
- find out the factors which determine the acceleration of a body rolling down an inclined plane without slipping and with no loss of energy due to friction
- learn the role of friction when a cylinder rolls down an inclined plane without slipping

6.2 Rolling Bodies

When an object with circular symmetry, as for instance, a sphere, a cylinder, a disc or a wheel, rolls on a plane surface its motion is a combination of translation and rotation. Consider the rolling motion of a circular object as shown in the Fig.6.1.

Rolling Motion



Translational + Rotational Motion

Play

Fig. 6.1 Motion of a circular disc rolling on a plane surface can be viewed as comprising of two parts: translational and rotational motions.

Notice that at any instant, the axis normal to the diagram through the point of contact P is the axis of rotation. Let the speed of the centre of mass be V (relative to the observer fixed on the surface). Then the instantaneous angular speed about the axis passing through P is ω=V/R, where R is the radius of the body. Therefore, at that instant, all the particles of the rigid body are moving with the same angular speed ω about the axis through P and the motion is pure rotation. [This, however, is not true about the linear speeds. For example, at an instant when the centre of mass is moving with linear speed V=R ω, the point P is at rest, point O has a speed V and the highest point of circumference has the speed 2V.]

Thus the kinetic energy(K.E) corresponding to pure rotation= $\frac{1}{2} I_P \omega^2$ (6.1)

where I_P represents the moment of inertia about the axis through P.

Using the theorem of parallel axis

$$I_P = I_{cm} + MR^2, \tag{6.2}$$

where I_{cm} is the moment of inertia of the body of mass M about a parallel axs passing through c.m O. Therefore, kinetic energy,

$$KE = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}MR^2\omega^2 = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}MV^2 \tag{6.3}$$

Thus, the kinetic energy of rolling motion can be expressed in two parts, one part corresponding to the rotational and the other to the translational motion.

6.2.1 Value Addition

Visit the following website to see an animation for free rolling of a circular object. Observe the velocity vectors of a point on the rim of the rolling body.

http://www.animations.physics.unsw.edu.au/jw/rolling.htm

http://www.phy.ntnu.edu.tw/ntnujava/index.php?topic=141.0

The following video demostrates the three cases when the velocity due to translation is less than, equal to and greater than the velocity of rotation and then illustrates the condition when the body rolls without slipping.

http://www.youtube.com/watch?v=s1qJrNfOCHs

<iframe width="560" height="315" src="//www.youtube.com/embed/s1qJrNfOCHs" frameborder="0" allowfullscreen></iframe>

Exercise

A solid sphere of mass 500 gm and radius 5 cm rolls without sliding with a uniform velocity of 10 cm/s along a straight line on a smooth horizontal table. Calculate its total energy.

Solution

$$\text{Total kinetic Energy} = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}MV^2 = \frac{1}{2}I_{cm}\frac{V^2}{R^2} + \frac{1}{2}MV^2$$

Now, given: Mass of the sphere=500gm; Radius, R=5cm; V=10 cm/s. We also know that moment of inertia of the sphere is $\frac{2}{5}MR^2$. The above expression reduces to

$$\frac{1}{2}MV^2\left(1+\frac{2}{5}\right) = \frac{1}{2}\times 0.5\times(10^{-2})^2\times\frac{7}{5} = 3.5\times 10^{-5}\ joules$$

6.3 Motion of a body Rolling down an inclined Plane

When a body rolls down an inclined plane without slipping, it acquires both translatory and rotatory motions. As the body rolls down, it looses its potential energy, due to vertical descent. It simultaneously acquires linear and angular velocities. As a result it gains kinetic energy of translation and rotation. If there is no loss of energy due to friction, the loss of potential energy must be equal to the gain in kinetic energy.

What are the factors which determine the acceleration of the boy rolling down an inclined plane? To find this out, let us consider a body of mass M and radius R rolling down a plane inclined at an angle θ to the horizontal (See Fig.6.2).

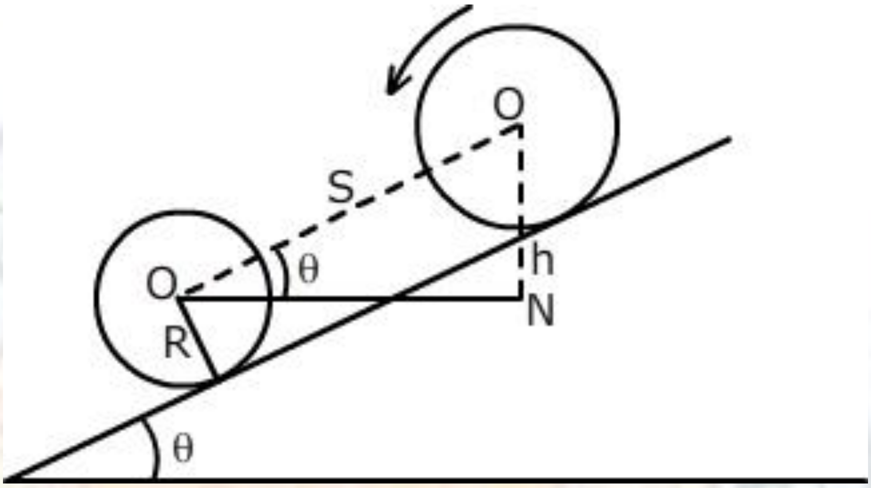


Fig. 6.2 A body rolling down an inclined plane

Let the body acquire angular speed ω and let linear speed acquired by the centre of mass after covering a distance s along the inclined plane be V.

Now loss of potential energy in travelling a vertical height, h, PE=M g h

Gain in kinetic energy (KE)= $\frac{1}{2}I_{CM}w^2 + \frac{1}{2}MV^2 = \frac{1}{2}Mk^2\frac{V^2}{R^2} + \frac{1}{2}MV^2 = \frac{1}{2}MV^2\left(\frac{k^2}{R^2}+1\right)$, (6.5)

where k is the radius of gyration about an axis through the centre of mass and parallel to the plane.

Applying the law of conservation of energy

$$Mgh = \frac{1}{2}MV^2\left(\frac{k^2}{R^2}+1\right) \qquad (6.6)$$

This gives

$$V^2 = \frac{2gh}{(1+k^2/R^2)} = \frac{2gs\ \sin(\theta)}{(1+k^2/R^2)}\,. \qquad (6.7)$$

the centre of mass of the body acquires an acceleration 'a' on covering the distance s, then $V^2 = 2as$. Using the relation Eq.(6.7), we get

$$a = \frac{g\ \sin(\theta)}{1 + k^2 / R^2} \qquad (6.8)$$

This Eq.(6.8) is an important one in so far as it tells us that for a given inclination, acceleration acquired by a rolling body is independent of its mass; it is inversely proportional to the factor $(1 + k^2 / R^2)$. For two bodies having identical shapes but of different moment of inertia, when allowed to roll down an inclined plane from the same height simultaneously, it is possible to find with the help of Eq.(6.8) which one would acquire greater acceleration and hence would reach the ground earlier.

Question Number	Type of question
1	Objective

Question

A ring, a circular disc and a sphere of the same radius and mass roll down an inclined plane from the same height h. Which of the three reaches the ground (i) first and (ii) last?

(a) ring reaches first and the disc last
(b) disc reaches first and the sphere last
(c) sphere reaches first and disc last
(d) sphere reaches first and the ring last

Correct choice	a) false b) False c) False d) True

Justification/ Feedback for the correct answer

The acceleration down an inclined plane of a rolling ring, disc and sphere are respectively (1/2) g sin θ, (2/3) g sin θ, (5/7) g sin θ

Note that the moment of inertia of the ring is $M\,R^2$, of disc is $(1/2)MR^2$ and of the sphere is $(2/5)MR^2$

Exercise

A solid disc (i) rolls (ii) slides from rest down an inclined plane. Neglecting friction, compare the velocities in the two cases when the disc reaches the bottom of incline.

Solution

The moment of inertia of the disc is $(1/2)MR^2$. Using the expression (6.8) for acceleration

$$a = \frac{g\,\sin(\theta)}{1 + k^2/R^2} = \frac{g\sin(\theta)}{1+1/2} = \frac{2g\sin(\theta)}{3}$$

If s is the distance of the inclined plane, then the velocity

$$V^2 = 2as = \frac{4g\,s\sin(\theta)}{3}.$$

In the case of sliding, the disc acquires the acceleration

$$a' = g\sin(\theta).$$

Therefore, $$V'^2 = 2g\sin(\theta)\times s$$

Comparing the two, we get $$\frac{V}{V'} = \sqrt{\frac{2}{3}}.$$

6.4 Rolling of a Cylinder without Slipping down an Inclined Plane

(**force of friction included**)

Let us describe the motion of a cylinder when it rolls down an inclined plane without slipping. **The condition for rolling without slipping is that at each instant, the line of contact is momentarily at rest and the cylinder is rotating about it as axis.**

Consider a cylinder of mass M, radius R rolling down a plane inclined at angle θ to the horizontal. The figure (Fig.(6.3)) shows a right cross section of the cylinder. What are the forces acting on the cylinder while rolling? These are:

(a) The weight M g of the cylinder acting vertically downward through the centre of mass,
(b) The normal force N acting at the point of contact P between the cylinder and the plane; and
(c) The force of friction f at P, acting tangentially upwards and parallel to the plane.

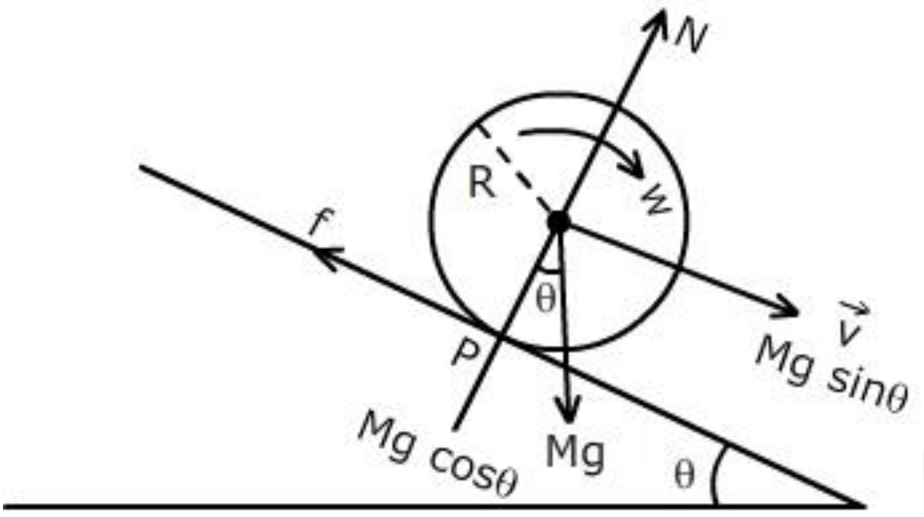


Fig. 6.3 Rolling of a cylinder without slipping down an inclined plane including the effect of friction.

Let the instantaneous angular velocity of rotation about an axis passing through P be ω, which is the same as for the rotation about the horizontal axis through the centre of mass. Then the linear velocity V and the acceleration a of the centre of mass of the cylinder rolling down the inclined plane are

$$V = R\,\omega \qquad \text{and} \qquad a = R\,\alpha\,, \tag{6.9}$$

where α is the angular acceleration of the cylinder down the plane.

Since there is no motion in a direction normal to the plane, we have

$$N = M\,g\cos(\theta) \tag{6.10}$$

Now, using Newton's second law for the linear motion of the centre of mass, the net force on the cylinder rolling down is

$$\begin{aligned} F &= M\ a = M\ \frac{dv}{dt} = M\,R\frac{d\omega}{dt} = M\,R\alpha \\ &= M\ g\sin\theta - f \end{aligned} \tag{6.11}$$

In terms of the torque T acting on the rolling cylinder we know

$$\begin{aligned} T &= \frac{dL}{dt} = I\frac{d\omega}{dt} = I\,\alpha \\ &= I\frac{a}{R} = R\,f \end{aligned} \quad , \tag{6.12}$$

where I is the moment of inertia of the solid cylinder about the axis of symmetry, through the centre of mass, given by $I = \frac{1}{2}M\,R^2$.

Note from Eq.(6.12), the torque T acting on the cylinder is being produced by the force of friction, i.e.,

$$f = \frac{I\,a}{R^2} \;, \qquad (6.13)$$

Using the relation $I = M\,k^2$ in Eq.(6.8), the linear acceleration 'a' of the rotating cylinder is given by

$$a = g\sin\theta - \frac{I\,a}{MR^2}$$

or $a = g\sin\theta/(1 + I/M\,R^2)$

Since $I/M\,R^2 = 1/2$, we get a=2(g sin(θ))/3 (6.14)

and f=(M g sin(θ))/3 (6.15)

From Eq.(6.14), it is clear that the linear acceleration 'a' of the solid cylinder rolling down the inclined plane is less than g- acceleration due to gravity. Also, . Eq.(6.15) shows that the force of friction f < Mg, i.e., the weight of the cylinder. Expressing the force of friction in terms of the coefficient of static friction,viz.,

$$f = \mu_S\,N \;,$$

and using Eq.(6.15) for f and Eq.(6.10) for N, we get

$$\mu_S = \frac{1}{3}\tan(\theta) \qquad (6.16)$$

The effect of moment of inertia on the linear acceleration a of the cylinder and the force of friction f can also be easily understood. Recall that if the cylinder were hollow, its moment of inertia would be, $I = M\,R^2$. The net effect on a and f would then be that, instead of the factors (2/3) and (1/3) appearing in Eqs.(6.14) and (6.15), we would simply have the factor of (1 / 2) in each of these equations.

6.4.1 Value Addition

It would be interesting to visit the following website for animations on rolling of bodies down an inclined plane and get some glimpses of Galileo's experiments on them

http://www.animations.physics.unsw.edu.au/jw/rotation.htm#rolling

http://www.youtube.com/watch?v=MAvPlHAfGbQ

http://www.phy.ntnu.edu.tw/ntnujava/index.php?topic=1025.msg3758#msg3758

http://physics-animations.com/Physics/English/angl_txt.htm

Rotational Dynamics / Mechanics-VI

http://www.youtube.com/watch?v=MAvPlHAfGbQ

Credit: By Dr. Michael R. Gallis
Penn State Schuylkill
Pennsylvania State University
mrg3@psu.edu

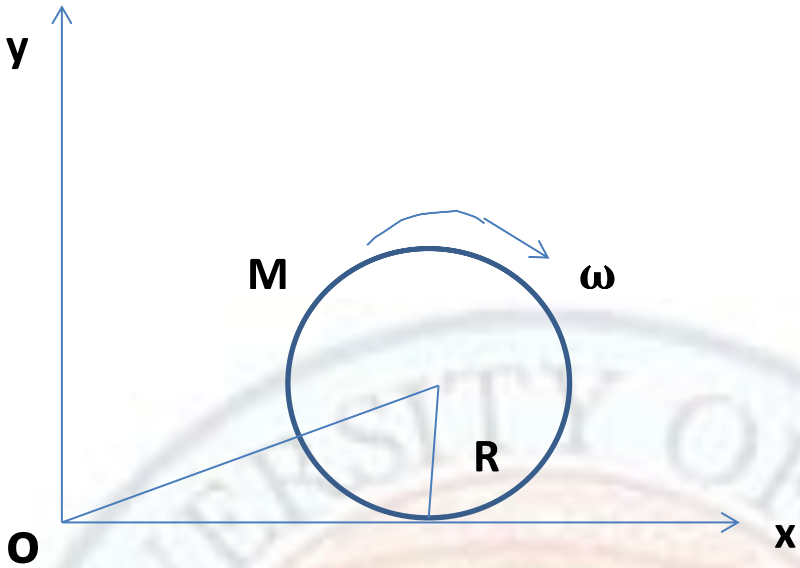
Summary

In this lesson, you learn

- the mechanics of rolling bodies on a plane surface
- to describe the motion of a body rolling down an inclined plane
- to analyze the rolling motion of a cylinder without slipping down an inclined plane (with force of friction)

Exercises/ Questions:

1. A disc of mass M is rolling with angular speed ω on a horizontal plane as shown in the figure.. What would be the magnitude of angular momentum of the disc about the origin?



Answer: Let c denote the centre of the disc. If L_C is the angular momentum of the didc about C and $p_C = M v_c$ is the linear momentum of the centre of mass of the disc, the angular momentum about the origin O is

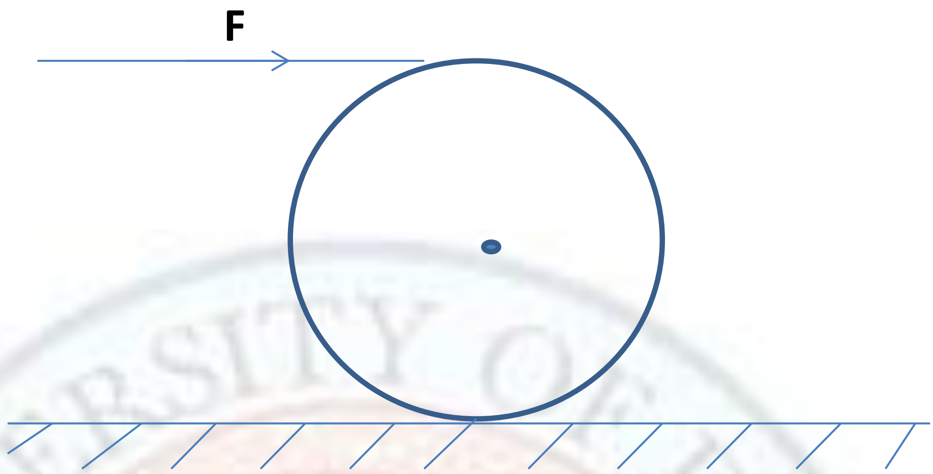
$$\vec{L}_O = \vec{L}_C + \vec{R}_C \times \vec{p}_C.$$ Its magnitude is given by

$$L_O = I_C\,\omega + R_C M v_C \sin\theta.$$

Now $I_C = \frac{1}{2}MR^2$ *and* $\sin\theta = R/R_C$ *and* $v_C = R\omega$.

Substituting these, we get $Lo = \frac{3}{2}MR^2\omega$

2. A hollow sphere of mass M and radius R is initially at rest on a horizontal rough surface. It moves under the action of a constant horizontal force F as shown in



the figure.

Does the frictional force between the sphere and the surface retard the motion of the sphere or make the sphere move faster?

Answer: If the force is applied above the centre of mass, the torque due to frictional force tends to rotate the sphere faster. Hence in this case, frictional force acts in the direction of motion and makes the sphere move faster.

3. In the question given above, what would be the linear acceleration of the sphere?

Answer: Let a and α be the linear and angular acceleration of the sphere. For translational motion F + f=M a (i), where f is the frictional force.

The magnitude of the net torque = F R – f R= I α =I a /R. For a hollow sphere,

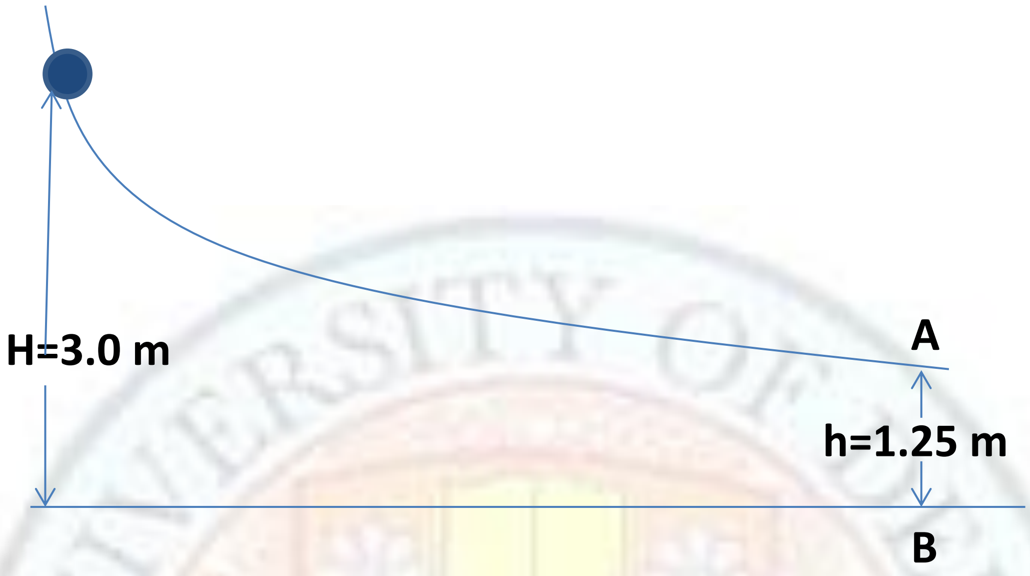
$I = \frac{2}{3}MR^2$. Therefore, F R-f R= $\frac{2}{3}MR^2 \times \frac{a}{R}$ giving F – f = 2 M a / 3 (ii).

From eqns. (i) and (ii), we get a= 6 F / 5 M.

4. In question 2, obtain the relation between the frictional force and the force applied.

Answer : From eqns. (i) and (ii) obtained above, we find f= M a / 6= F/5.

5. A small sphere rolls down without slipping from the top of a track in a vertical plane. The track has an elevated section and a horizontal part. The horizontal part is 0.7 m above the ground and the top of the track is 3.5 m above the ground (see figure). Taking the value of g=10 m / s^2 , find the horizontal speed when the sphere reaches point A.



Answer: The loss in potential energy when the sphere moves from the top of the track to the point A = gain in total (translational+ rotational) kinetic energy, i.e.,

$$Mg(H-h)=\frac{1}{2}Mv^2+\frac{1}{2}I\omega^2 \text{. Now } I=\frac{2}{5}MR^2 \text{ and } \omega=v/R.$$

This gives $Mg(H-h)=\frac{1}{2}Mv^2+\frac{1}{5}Mv^2=\frac{7}{10}Mv^2 \Rightarrow v=\left[\frac{10(H-h)g}{7}\right]^{1/2}$

Substituting the values of g, H and h, we get v=5.0 m/ s.

6. In the above question, what would be the time taken by the sphere to fall through h=1.25 m?

Answer: The time of flight, $t=\sqrt{\frac{2h}{g}}=0.5s$.

7. In question 5, find the distance covered on the ground with respect to the point B.

Answer: horizontal distance covered is v t=5.0×0.5=2.5 m.

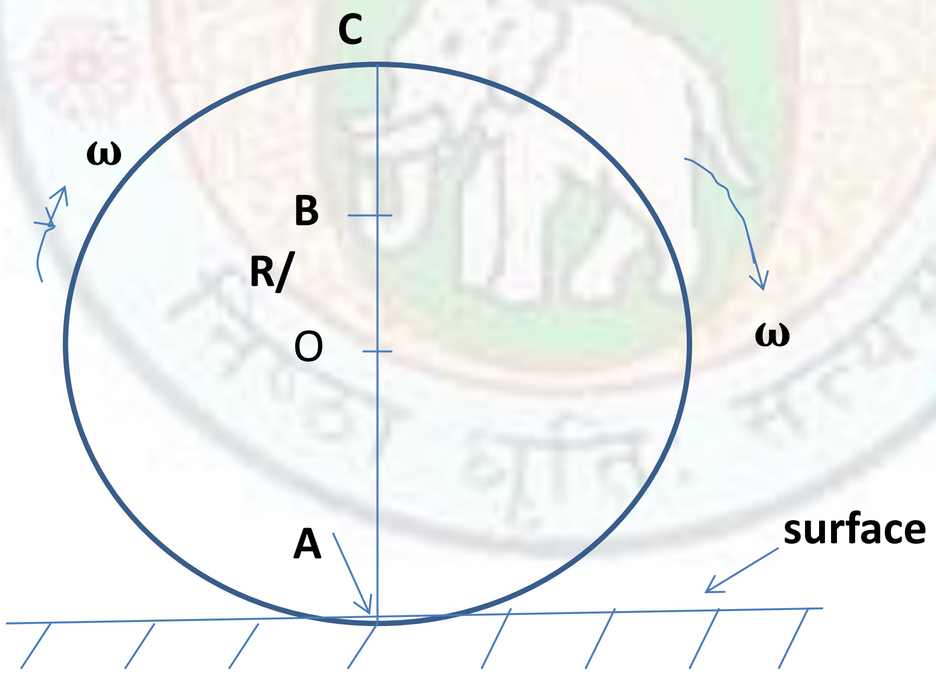
8. In Q.5, after the sphere leaves the point A, during its motion as a projectile, would it stop rotating or continue to rotate as about its centre of mass?

Answer: During its flight as a projectile from point A to the point it hits the ground, since there is no external torque acting on it, the angular momentum remains unchanged and therefore the sphere will continue to rotate about the centre of mass.

9. A uniform disc of radius R is rolling (without slipping) on a horizontal surface with an angular speed ω. As shown in the figure, points A and C are located on the rim and point B is at a distance R/2 from the centre O. During rolling, the points A, B, and C lie on the vertical diameter at a certain instant of time. If v_A, , v_B *and* v_C are the linear speeds of points A, B and C respectively, then which one of the following is correct:

(1) $v_A = v_B = v_C$ (2) $v_A \rangle v_B \rangle v_C$ (3) $v_A = 0,\ v_C = \frac{4}{3} v_B$ or

(4) $v_A = 0,\ v_C = 2\, v_B$



Answer: The disc is rolling about the point O. Thus the axis of rotation passes through the point A and is perpendicular to the plane of the disc. From the relation v=r ω, r=0. Therefore, $v_A=0$, $v_B=(AB)\omega=\frac{3}{2}R\omega$, $v_C=2R\omega=\frac{4}{3}v_B$.

Correct choice is (3).

10. A sphere rolls down an inclined plane without slipping. What fraction of its total energy is rotational?

Answer: Rotational kinetic energy, ER= $I\,\omega^2/2=\frac{1}{2}(\frac{2}{5}MR^2)\omega^2$

Translational Kinetic Energy $=\frac{1}{2}M\,v^2=\frac{1}{2}MR^2\omega^2$

Total Energy= Rotational +Translational= $=\frac{7}{10}MR^2\omega^2$

Rotational Kinetic Energy=(2/7) of total kinetic energy