Discipline Course-I
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Paper: Mathematical PhysicsI IA
Lesson: Second Order Linear Homogeneous Differential Equation
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Chapter 6: Second Order Linear Homogeneous Differential Equation

Learning Objectives

In this chapter the student will be able to learn

- ***what are General Second Order 'Linear Homogeneous' Equations?***
- ***few properties of solutions of such DE; the trivial solution and complementary functions.***
- ***linear dependence and independence of the complementary solutions.***
- ***how to determine the solution of Second Order 'Linear Homogeneous' Equations?***
- ***Theorem of Superposition of Solution to Second Order LDE***
- ***Initial Value Solution of Second Order Linear DE***
- ***Basis General Solution and Particular Solution of Second Order Linear DE Second Order Linear Homogeneous DE with Constant Coefficients; the Auxiliary Equation of the DE and its roots; nature of roots and the corresponding solutions.***

Second Order Linear Homogeneous DE

6.1 General Second Order 'Linear Homogeneous' Equations

A general second order linear differential equation in $y(x)$ will be

$$A(x)y'' + B(x)y' + C(x)y + D(x) = 0$$

where the coefficients $A(x), B(x), C(x)$ & $D(x)$ are functions of the independent variable x alone or constants. We can divide by $A(x)$ (in the interval of x for which $A(x) \neq 0$) and rewrite the *DE* in the Normal Form or Standard Form as

$$y'' + p(x)y' + q(x)y = r(x)$$

where

$$p(x) = B(x)/A(x)$$
$$q(x) = C(x)/A(x) \;\&$$
$$r(x) = -D(x)/A(x)$$

The equation is linear in the unknown function y & its derivative y''& y'. If the equation cannot be written in this form then the equation is non-linear. A **nonlinear** equation will contain product terms of the type $y\,y'$, $y'y''$ etc.

As before, if the RHS of *Equation 1* is zero i.e., $r(x) = 0$ the equation is said to be *Homogeneous or Reduced* otherwise it is said to be *Non-Homogeneous*.

We would like to note few properties of such *DE*

a) *The solution $y(x) = 0$ is a particular solution of the Homogeneous Equation and is called the trivial solution*.

b) *If $y_{cf}(x)$ is a non-zero solution (known as the Complementary Function) of the Homogeneous Equation then $Ay_{cf}(x)$ is also a solution to the Homogeneous equation where A is an arbitrary constant.*

c) *If the Homogeneous Equation admits two Complementary Functions $y_{1,cf}(x)$ & $y_{2,cf}(x)$ then $C_1y_{1,cf}(x) + C_2y_{2,cf}(x)$ is also a solution to the Homogeneous equation where C_1 & C_2 are some arbitrary constants.*

d) *If $y_{1,cf}(x)$ & $y_{2,cf}(x)$ are Linearly Independent solutions ($i.e., C_1y_{1,cf}(x) + C_2y_{2,cf}(x) = 0$ on an interval I if and only if $C_1 = C_2 = 0$) of the Homogeneous Equation then every solution to the Homogeneous equation is given by the composite*

$$y_{cf}(x) = C_1y_{1,cf}(x) + C_2y_{2,cf}(x)$$

where C_1 & C_2 are arbitrary constants.

6.2 Determination of Solution of Second Order Linear DE

A solution of the second order (linear or nonlinear) *DE* on some open interval $a < x < b$ is function $y_{pi}(x)$ that has defined derivatives $y_{pi}'(x)$ and $y_{pi}''(x)$ and satisfies that *DE* for all x in that interval. In other words the *DE* becomes an identity if we replace the unknown function $y(x)$ & its derivatives by $y_{pi}(x)$ and its derivatives $y_{pi}'(x)$ & $y_{pi}''(x)$.

6.2.1 Theorem of Superposition of Solution to Second Order LDE

Theorem 1

For the Second Order Linear Homogeneous DE

$$y'' + p(x)y' + q(x)y = 0$$

any Linear combination of two solutions on an open interval I is again a solution on I.

Proof

Consider $y_{1,cf}(x)$ & $y_{2,cf}(x)$ as two solution of the *Homogeneous DE* on an interval I. Let

$$y_{cf} = C_1 y_{1,cf}(x) + C_2 y_{2,cf}(x)$$

where C_1 & C_2 are arbitrary constants.
Now the derivatives of this would be

$$y_{cf}' = C_1 y_{1,cf}'(x) + C_2 y_{2,cf}'(x)$$
$$y_{cf}'' = C_1 y_{1,cf}''(x) + C_2 y_{2,cf}''(x)$$

and upon substitution we get

$$y_{cf}'' + p(x)y_{cf}' + q(x)y_{cf}$$
$$= [C_1 y_{1,cf}''(x) + C_2 y_{2,cf}''(x)] + p(x)[C_1 y_{1,cf}'(x) + C_2 y_{2,cf}'(x)] + q(x)[C_1 y_{1,cf}(x) + C_2 y_{2,cf}(x)]$$
$$= C_1[y_{1,cf}'' + p(x)y_{1,cf}' + q(x)y_{1,cf}] + C_2[y_{2,cf}'' + p(x)y_{2,cf}' + q(x)y_{2,cf}] = 0$$

as the expressions in the bracket are zero since we have assumed that $y_{1,cf}$ & $y_{2,cf}$ were the solutions of the *Homogeneous DE* on an interval I. Hence

$$y_{cf}'' + p(x)y_{cf}' + q(x)y_{cf} = 0$$

and y_{cf} too is a solution to the *Homogeneous DE* on an interval I.
Conversely, let

$$y_{cf} = C_1 y_{1,cf}(x) + C_2 y_{2,cf}(x)$$

is a solution to the *Homogeneous DE* on an interval I.
Now upon substitution we get

$$y_{cf}'' + p(x)y_{cf}' + q(x)y_{cf} = 0$$
$$[C_1 y_{1,cf}''(x) + C_2 y_{2,cf}''(x)] + p(x)[C_1 y_{1,cf}'(x) + C_2 y_{2,cf}'(x)] + q(x)[C_1 y_{1,cf}(x) + C_2 y_{2,cf}(x)] = 0$$
$$C_1[y_{1,cf}'' + p(x)y_{1,cf}' + q(x)y_{1,cf}] + C_2[y_{2,cf}'' + p(x)y_{2,cf}' + q(x)y_{2,cf}] = 0$$

as the coefficients C_1 & C_2 are essentially non-zero therefore we must have $y_{1,cf}$ & $y_{2,cf}$ as the solutions of the *Homogeneous DE* on an interval I so that LHS reduces to zero. Hence,

$$y_{1,cf}'' + p(x)y_{1,cf}' + q(x)y_{1,cf} = 0$$
$$y_{2,cf}'' + p(x)y_{2,cf}' + q(x)y_{2,cf} = 0$$

and $y_{1,cf}$ & $y_{2,cf}$ too are solutions to the *Homogeneous DE* on an interval I.

This theorem represents the Principle of Superposition and, should be noted, does not hold for Non-Homogeneous / Non-Linear DE.

6.2.2 Value Addition Examples

Example 6.2.2.1 A *Non-Homogeneous DE* does not admit a *Linear Combination* y_{lc} of its solutions y_1 & y_2 to be a solution itself. For this consider the following *DE*

$$\boldsymbol{y'' + y = a}$$

which admits two solutions $y_1 = a + \cos x$ & $y_2 = a + \sin x$.

Solution:
Let's first check whether $y_1 = a + \cos x$ & $y_2 = a + \sin x$ are solution to the given *DE*. We find

$$y_1' = -\sin x$$
$$y_2' = \cos x$$

$$y_1'' = -\cos x$$
$$y_2'' = -\sin x$$

Therefore

$$y_1'' + y_1 = -\cos x + (a + \cos x) = a$$
$$y_2'' + y_2 = -\sin x + (a + \sin x) = a$$

and so y_1 & y_2 are indeed the solutions.
We now test a linear combination of the two solutions

$$y_{lc} = C_1 y_1 + C_2 y_2$$
$$y_{lc}' = C_1 y_1' + C_2 y_2'$$
$$y_{lc}'' = C_1 y_1'' + C_2 y_2''$$

And so

$$y_{lc}'' + y_{lc} = (C_1 y_1'' + C_2 y_2'') + (C_1 y_1 + C_2 y_2)$$
$$y_{lc}'' + y_{lc} = C_1(y_1'' + y_1) + C_2(y_2'' + y_2)$$

Now substituting the values on the RHS $y_1'' + y_1 = a$ & $y_2'' + y_2 = a$

$$y_{lc}'' + y_{lc} = (C_1 + C_2)a$$
$$y_{lc}'' + y_{lc} \neq a$$

which shows y_{lc} is not a solution to the given *DE*.

Example 6.2.2.2 A *Non-Linear DE* does not admit a *Linear Combination* y_{lc} of its solutions y_1 & y_2 to be a solution itself. For this consider the following *DE*

$$yy'' + xy' = 0$$

which admit two solutions $y_1 = -x^2$ & $y_2 = a$. Show that $y_{lc} = 1 \times a + a \times (-x^2) = a(1 - x^2)$ and $y_{lc} = 1 \times a + (-a) \times (-x^2) = a(1 + x^2)$ are not solutions.

Solution:
Let's first check whether $y_1 = -x^2$ & $y_2 = a$ are solution to the given *DE*. We find

$$y_1{}' = -2x$$
$$y_2{}' = 0$$

$$y_1{}'' = -2$$
$$y_2{}'' = 0$$

Therefore

$$y_1 y_1{}'' + xy_1' = (-x^2) \times (-2) + x \times (-2x) = 0$$
$$y_2 y_2{}'' + xy_2' = (-x^2) \times 0 + x \times 0 = 0$$

and so y_1 & y_2 are indeed the solutions.
We now test a linear combination of the two solutions

$$y_{lc} = a(1 - x^2)$$
$$y_{lc}{}' = a(0 - 2x)$$
$$y_{lc}{}'' = -2a$$

And so

$$y_{lc} y_{lc}'' + xy_{lc}' = a(1 - x^2) \times (-2a) + x(-2ax) = -2a^2 + 2a^2x^2 - 2ax^2 \neq 0$$

also

$$y_{lc} y_{lc}'' + xy_{lc}' = a(1 + x^2) \times (2a) + x(2ax) = 2a^2 + 2a^2x^2 + 2ax^2 \neq 0$$

which shows given y_{lc} are not a solution to the given *DE*. This would be the case for any linear combination y_{lc}.

6.3 Initial value Solution of Second Order Linear DE

We know that for a second order homogeneous *DE*

$$y'' + p(x)y' + q(x)y = 0$$

a general solution will be any linear combination of any two independent solutions $y_{1,cf}(x)$ & $y_{2,cf}(x)$ with arbitrary constants C_1 & C_2 as

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

The initial value problem now consists of two initial conditions

$$y(a) = b_0$$
$$y'(a) = b_1$$

which at some given point $x = a$ in the open interval have the solution and its derivative defined.

6.4 Basis General Solution and Particular Solution of Second Order Linear DE

If the two solutions $y_{1,cf}(x)\,\&\,y_{2,cf}(x)$ are linearly independent i.e., we can say that on an interval I when the solutions are defined then

$$C_1y_1(x) + C_2y_2(x) = 0$$

on I implies that $C_1, C_2 = 0$ then a pair $y_{1,cf}(x)\,\&\,y_{2,cf}(x)$ of linearly independent solutions form a Basis (or fundamental system) on the interval I.

We define linear dependency if we can express

$$y_1(x) = -\left(\frac{C_2}{C_1}\right)y_2(x) \text{ or } y_2(x) = -\left(\frac{C_1}{C_2}\right)y_1(x)$$

i.e., when $y_1\,\&\,y_2$ are proportional.

Example 6.4.1 Verify that for the *Linear DE*

$$\boldsymbol{y'' + \alpha^2 y = 0}$$

the functions $\cos\alpha x$ and $\sin\alpha x$ form a Basis of solutions. Also find the particular solution that yields $y(0) = 4$ and $y'(0) = -6\alpha$.

Solution:
Let's first check whether $y_1 = \cos\alpha x\ \&\,y_2 = \sin\alpha x$ are solution to the given *DE*. We find

$$y_1{}' = -\alpha\sin\alpha x$$
$$y_2{}' = \alpha\cos\alpha x$$

$$y_1{}'' = -\alpha^2\cos\alpha x$$
$$y_2{}'' = -\alpha^2\sin\alpha x$$

Therefore

$$y_1{}'' + \alpha^2 y_1 = -\alpha^2\cos\alpha x + \alpha^2 \times \cos\alpha x = 0$$
$$y_1{}'' + \alpha^2 y_1 = -\alpha^2\sin\alpha x + \alpha^2 \times \sin\alpha x = 0$$

and so $y_1\,\&\,y_2$ are indeed the solutions.

We now test a linear independence of the two solutions

$$ay_1(x) + by_2(x) = 0$$
$$a\cos\alpha x + b\sin\alpha x = 0$$
$$b\tan\alpha x = -a$$

where the *LHS* is a variable while the *RHS* is an arbitrary constant. This gives no choice but to have $b = a = 0$.

Now that we have tested that y_1 & y_2 indeed forms the *Basis*. The general solution would be

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$
$$y(x) = C_1\cos\alpha x + C_2\sin\alpha x$$

There will be a family of curve depending on the values of C_1 & C_2. Also each member of the family will have unique derivative

$$y'(x) = \alpha(-C_1\sin\alpha x + C_2\cos\alpha x)$$

The initial value problem is to find the unique member-curve which has $y = 4$ and $y' = -6\alpha$ at $x = 0$. We thus find

$$4 = y(x = 0) = a \times 1 + b \times 0 = a$$
$$-6\alpha = y'(x = 0) = -\alpha a \times 0 + \alpha b \times 1 = \alpha b$$

which requires

$$a = 4\ \&\ b = -6$$

Therefore, the particular solution would be

$$y(x) = 4\cos\alpha x - 6\sin\alpha x$$

6.5 To obtain the Basis if One Solution for the Second Order Linear Homogeneous DE is known

If a solution $y_1(x)$ is known then to obtain the *General Solution* of the Second Order Linear Homogeneous *DE* we need a second solution $y_2(x)$ which is linearly independent of $y_1(x)$ on an interval I when the solutions are defined. For this we introduce the *method of Reduction of Order*.

Let $u(x)$ be a function such that if $y_1(x)$ is a solution then $y_2(x) = u(x)y_1(x)$ be a linearly independent solution. We then have

$$y_2' = uy_1' + u'y_1$$
$$y_2'' = (uy_1'' + u'y_1') + (u''y_1 + u'y_1') = uy_1'' + u''y_1 + 2u'y_1'$$

Substituting these in the *DE*, $y'' + p(x)y' + q(x)y = 0$, we get

$$(uy_1'' + u''y_1 + 2u'y_1') + p \times (uy_1' + u'y_1) + q \times uy_1 = 0$$

Separately collecting terms of u'', u' & u we find

$$(u''y_1) + (2u'y_1' + pu'y_1) + (uy_1'' + puy_1' + quy_1) = 0$$
$$y_1u'' + (2y_1' + py_1)u' + (y_1'' + py_1' + qy_1)u = 0$$
$$y_1u'' + (2y_1' + py_1)u' + 0 \times u = 0$$

since y_1 is known to be a solution of the *DE* and satisfies $y_1'' + py_1' + qy_1 = 0$.

Therefore, we arrive at

$$y_1u'' + (2y_1' + py_1)u' = 0$$

which is linear first order in the variable u'. If y_1 & u' does not vanish in the interval I then we can write

$$\frac{u''}{u'} = -\left(2\frac{y_1'}{y_1} + p\right)$$
$$\frac{du'}{u'} = -\left(2\frac{dy_1}{y_1} + pdx\right)$$
$$\ln u' = -2\ln y_1 - \int pdx$$
$$u' = \frac{1}{{y_1}^2}e^{-\int pdx}$$
$$u = \int dx\left\{\frac{1}{{y_1}^2}e^{-\int pdx}\right\}$$

The second solution would hence be

$$y_2 = uy_1 = y_1\int dx\left\{\frac{1}{{y_1}^2}e^{-\int pdx}\right\}$$

6.6 Second Order Linear Homogeneous DE with Constant Coefficients

A Second Order Linear Homogeneous *DE* with constant coefficients written as

$$y'' + ay' + by = 0$$

has important applications in the field of physics and electrical circuits.

A likely choice of the solution is of the form $e^{\lambda x}$ (where λ is some constant) since all its derivative of any order is linearly proportional to $e^{\lambda x}$. This would then allow *LHS* under some conditions on λ.
So, we choose $y = e^{\lambda x}$ so that $y' = \lambda e^{\lambda x} = \lambda y$ & $y'' = \lambda^2 e^{\lambda x} = \lambda^2 y$ and

$$y'' + ay' + by = 0$$
$$\lambda^2 y + a\lambda y + by = 0$$
$$(\lambda^2 + a\lambda + b)y = 0$$

If y were to be a non-zero solution of the *DE* then λ must satisfy the equation

$$\lambda^2 + a\lambda + b = 0$$

This is called the *Auxiliary (Characteristics) Equation* and is quadratic in λ. Thus there will be two values of λ for which $y = e^{\lambda x}$ will be solution to the *DE*. If we denote the two roots as λ_1 & λ_2 then $e^{\lambda_1 x}$ & $e^{\lambda_2 x}$ will be two linearly independent solutions of the *DE*.
Now let's find the roots of the auxiliary equation

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4 \times 1 \times b}}{2 \times 1} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Depending on the sign of the discriminant we have the following cases

Case 1:

If $a^2 - 4b > 0$ then the two roots λ_1 & λ_2 are real and distinct ($\lambda_1 \neq \lambda_2$) and $e^{\lambda_1 x}$ & $e^{\lambda_2 x}$ will be the two linearly independent solutions forming the Basis of the DE.

Case 2:

If $a^2 - 4b = 0$ then the two roots λ_1 & λ_2 are real but equal ($\lambda_1 = \lambda_2 = \lambda$) and $e^{\lambda x}$ & $xe^{\lambda x}$ will be the two linearly independent solutions forming the Basis of the DE.

Case 3:

If $a^2 - 4b < 0$ then the two roots λ_1 & λ_2 are complex but conjugate ($\lambda_1 = \alpha + i\beta$ & $\lambda_2 = \alpha - i\beta$) and $e^{\alpha x} \cos \beta x$ & $e^{\alpha x} \sin \beta x$ will be the two linearly independent solutions forming the Basis of the DE.

We discuss these cases separately:

6.6.1 Case 1: Two real but distinct roots λ_1 & λ_2

Here in this case we have $y_1 = e^{\lambda_1 x}$ & $y_1 = e^{\lambda_2 x}$ as the two roots for the *DE* given by

$$y'' + ay' + by = 0$$

and

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \;\&\; \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

We find that

$$\frac{y_1(x)}{y_2(x)} = e^{(\lambda_1 - \lambda_2)x} = e^{\left(\sqrt{a^2-4b}\right)x}$$

is not a constant and so y_1 & y_2 are not linearly proportional to one another. Hence, they form a Basis of the solutions of the *DE*. The general solution would then be

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

$$y = C_1 e^{\frac{-a+\sqrt{a^2-4b}}{2}x} + C_2 e^{\frac{-a-\sqrt{a^2-4b}}{2}x}$$

$$y = e^{-\frac{a}{2}x}\left\{C_1 e^{+\frac{\sqrt{a^2-4b}}{2}x} + C_2 e^{-\frac{\sqrt{a^2-4b}}{2}x}\right\}$$

since $a > \sqrt{a^2 - 4b}$ the all such solution decay with increasing x.

Example 6.6.1.1 Solve the initial value problem for the *Linear Homogeneous DE*

$$y'' - y' - 2y = 0, \qquad y(0) = 3, \qquad y'(0) = 0$$

Solution:

Step 1 The Auxiliary Equation for the given *DE* will be obtained by replacing $y'' \to \lambda^2$, $y' \to \lambda$ & $y \to 1$

$$\lambda^2 - \lambda - 2 = 0$$

The roots are then found as

$$\lambda_1 = \frac{-(-1) + \sqrt{(-1)^2 - 4(-2)}}{2} \;\&\; \lambda_2 = \frac{-(-1) - \sqrt{(-1)^2 - 4(-2)}}{2}$$

$$\lambda_1 = \frac{1 + \sqrt{9}}{2} = 2 \;\&\; \lambda_2 = \frac{1 - \sqrt{9}}{2} = -1$$

Step 2 The General Solution would be

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$
$$y = C_1 e^{2x} + C_2 e^{-x}$$

Step 3 The Particular Solution satisfying the initial conditions require

$$y(0) = [C_1 e^{2x} + C_2 e^{-x}]_{x=0} = 3$$

$$C_1 + C_2 = 3$$

and

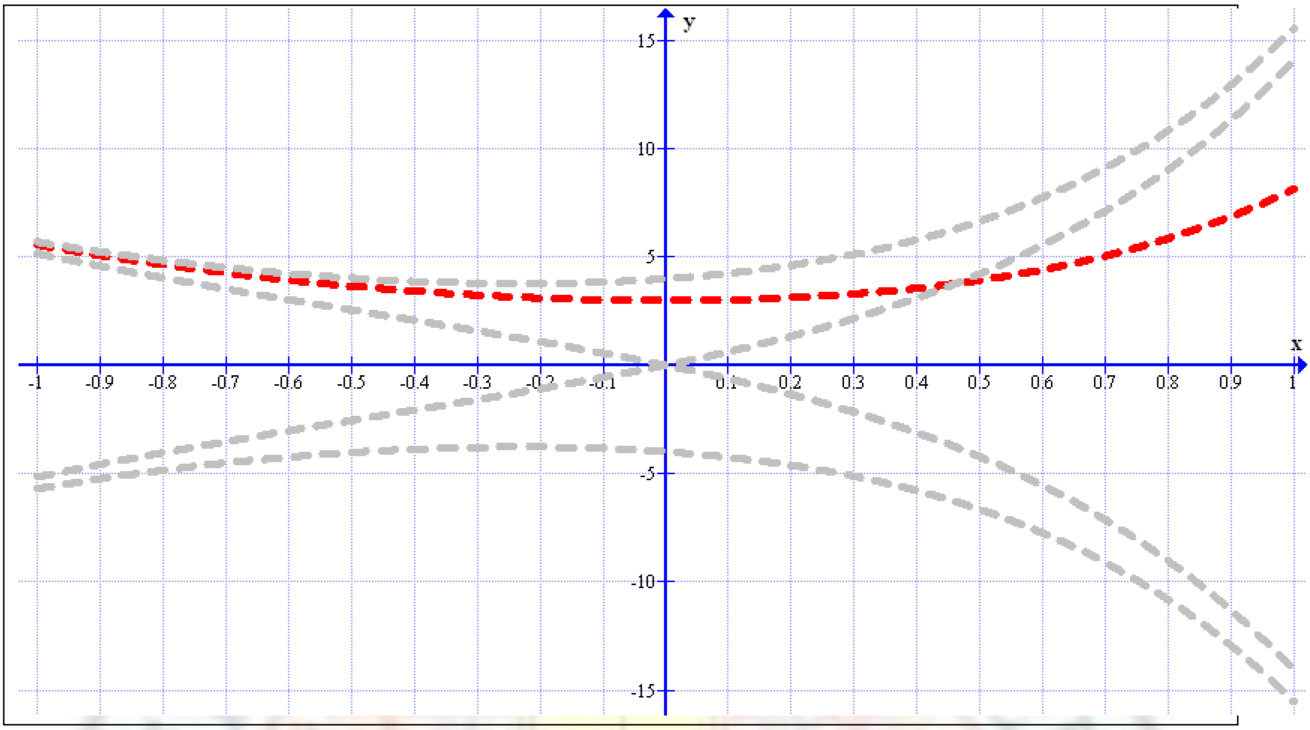
$$y'(0) = [2C_1 e^{2x} - C_2 e^{-x}]_{x=0} = 0$$

$$2C_1 - C_2 = 0$$

On solving we get $C_1 = 1$ & $C_2 = 2$ so that the particular solution comes out to be

$$y = e^{2x} + 2e^{-x}$$

Let's see it on the graph:



Example 6.6.1.2 **Solve the initial value problem the *Linear Homogeneous DE***

$$\boldsymbol{y'' - 4y = 0, \qquad y(0) = 3, \qquad y'(0) = 0}$$

Solution:

Step 1 The Auxiliary Equation for the given DE will be obtained by replacing $y'' \to \lambda^2$, $y' \to \lambda$ & $y \to 1$

$$\lambda^2 - 4 = 0$$

The roots are then found as

$$\lambda_1 = +\sqrt{4} = 2 \;\&\; \lambda_2 = -\sqrt{4} = -2$$

Step 2 The General Solution would be

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$
$$y = C_1 e^{2x} + C_2 e^{-2x}$$

Step 3 The Particular Solution satisfying the initial conditions require

$$y(0) = [C_1 e^{2x} + C_2 e^{-2x}]_{x=0} = 3$$
$$C_1 + C_2 = 3$$

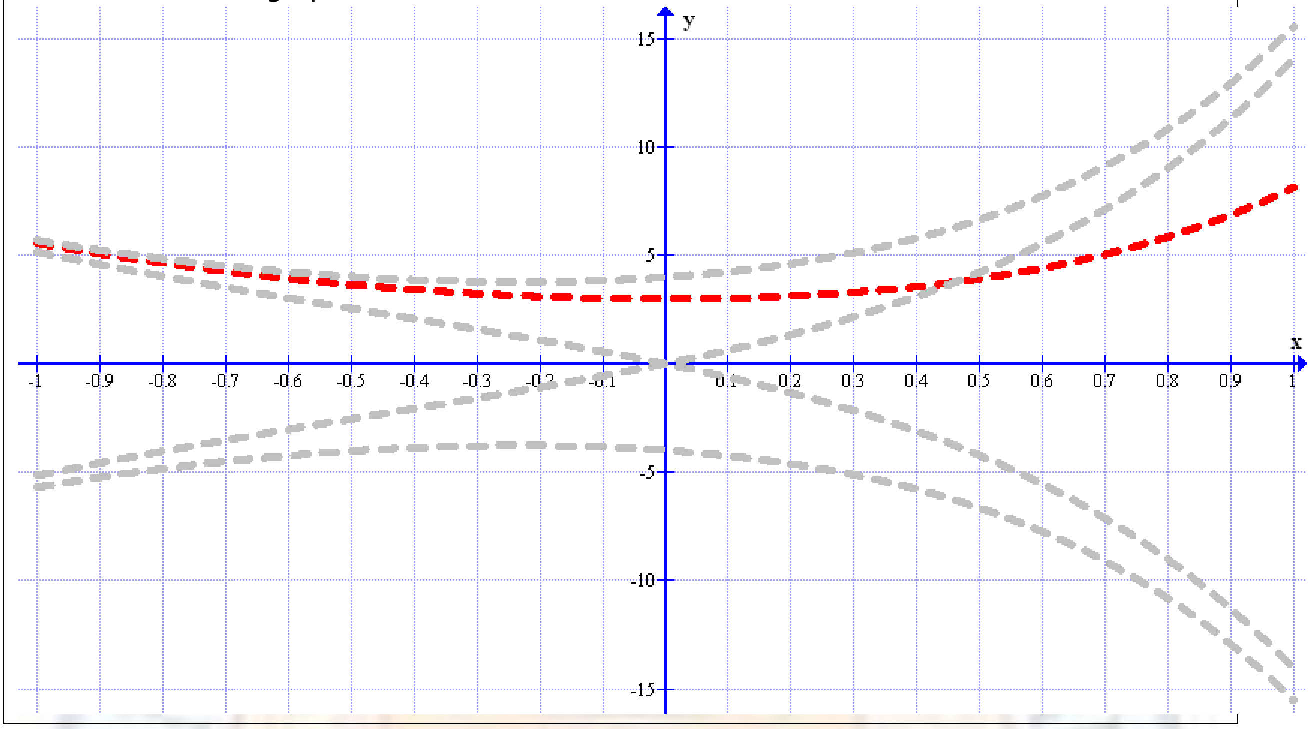
and

$$y'(0) = [2C_1 e^{2x} - 2C_2 e^{-2x}]_{x=0} = 0$$
$$2C_1 - 2C_2 = 0$$
$$C_1 = C_2$$

On solving we get $C_1 = 3/2$ & $C_2 = 3/2$ so that the particular solution comes out to be

$$y = \frac{3}{2}(e^{2x} + e^{-2x})$$

Let's see it on the graph:



6.6.2 Case 2: Two real but equal roots $\lambda_1 = \lambda_2 = \lambda$

In such case we have

$$\lambda_1 = -\frac{a}{2} = \lambda_2$$

so the only unique solution found for the *DE* is $y_1 = e^{\lambda x}$. *We need to find the other linearly independent solution for the DE.* As discussed before, let's assume second independent solution to be

$$y_2(x) = u(x)y_1(x)$$

for some appropriate function $u(x)$.

If y_1 & uy_1 were to be that solution of the *DE* $y'' + ay' + by = 0$ then as found earlier we must satisfy the relation,

$$y_1 u'' + (2y_1' + ay_1)u' = 0$$

Since $y_1 = e^{\lambda x}$ we have

$$e^{\lambda x}u'' + \left(2\lambda e^{\lambda x} + ae^{\lambda x}\right)u' = 0$$
$$u'' + (2\lambda + a)u' = 0$$
$$\frac{u''}{u'} = -(2\lambda + a)$$
$$\frac{u''}{u'} = -\left\{2\left(-\frac{a}{2}\right) + a\right\} = 0$$

Thus,

$$\frac{d}{dx}\ln u' = 0$$

$$u' = A$$

Therefore, we eventually have

$$u = Ax + B$$

and

$$y_2(x) = (Ax + B)e^{\lambda x}$$

Note again that y_1 & y_2 are not linearly proportional to one other. Hence, they form a Basis of the solutions of the *DE*. The general solution would then be, for any arbitrary a_1 & a_1,

$$y = a_1 y_1(x) + a_2 y_2(x)$$
$$y = a_1 e^{\lambda x} + a_2(Ax + B)e^{\lambda x}$$
$$y = (Aa_2 x + Ba_2 + a_1)e^{\lambda x}$$

By putting $C_1 = Aa_2$ & $C_2 = Ba_2 + a_1$

$$y = (C_1 x + C_2)e^{-\frac{a}{2}x}$$

and all such solutions decay with increasing x.

Example 6.6.2.1 Solve the initial value problem for the *Linear Homogeneous DE*

$$\boldsymbol{y'' - 4y' + 4y = 0, \qquad y(0) = 3, \qquad y'(0) = 4}$$

Solution:
Step 1 The Auxiliary Equation for the given *DE* will be obtained by replacing $y'' \to \lambda^2$, $y' \to \lambda$ & $y \to 1$

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots are then found as

$$\lambda_1 = \frac{-(-4) + \sqrt{(-4)^2 - 4(4)}}{2} \;\&\; \lambda_2 = \frac{-(-4) - \sqrt{(-4)^2 - 4(4)}}{2}$$
$$\lambda_1 = \frac{4 + \sqrt{0}}{2} = 2 \;\&\; \lambda_2 = \frac{4 - \sqrt{0}}{2} = 2$$

The roots are equal $\lambda_1 = \lambda_2 = 2$.

Step 2 The General Solution would be

$$y = (C_1 x + C_2)e^{2x}$$

Step 3 The Particular Solution satisfying the initial conditions require

$$y(0) = [(C_1 x + C_2)e^{2x}]_{x=0} = 3$$

$$C_2 = 3$$

and

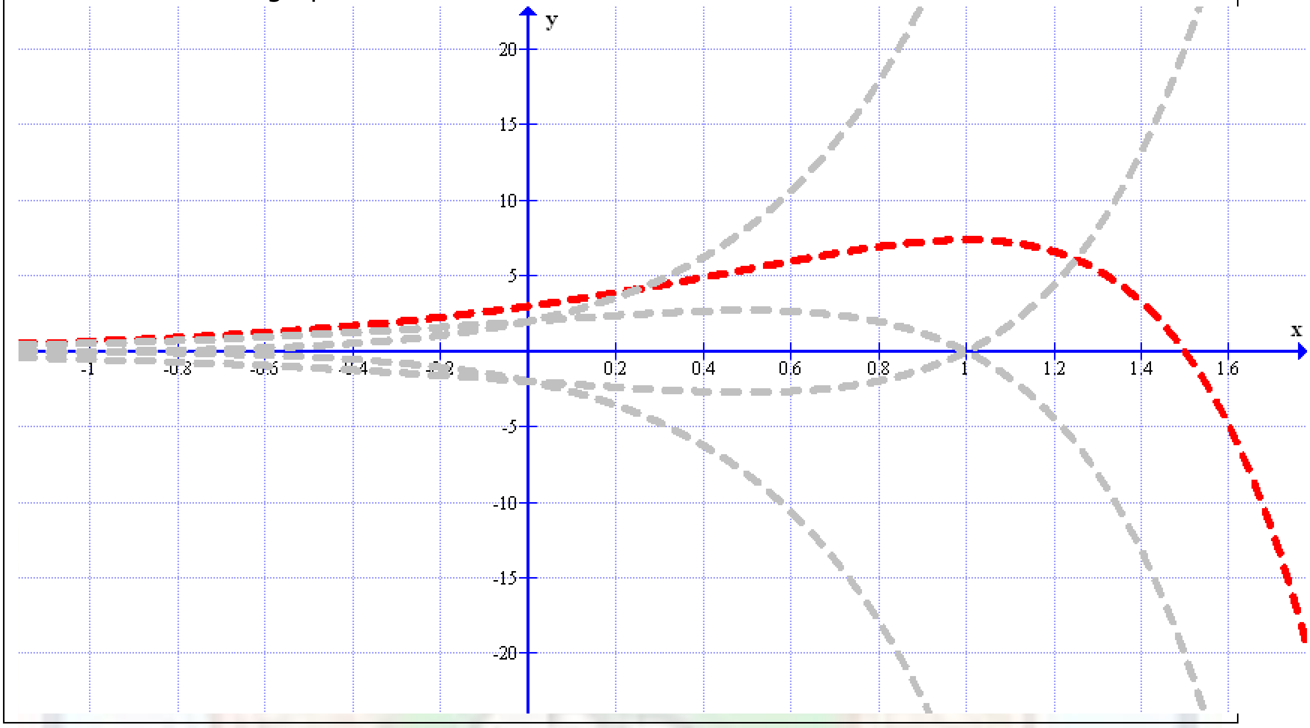
$$y'(0) = [C_1 e^{2x} + 2(C_1 x + C_2)e^{2x}]_{x=0} = 4$$

$$C_1 + 2C_2 = 4$$

On solving we get $C_1 = -2$ & $C_2 = 3$ so that the particular solution comes out to be

$$y = (-2x + 3)e^{2x}$$

Let's see it on the graph:



Example 6.6.2.2 Solve the initial value problem the *Linear Homogeneous DE*

$$\boldsymbol{y'' + 4y' + 4y = 0, \qquad y(0) = 3, \qquad y'(0) = 4}$$

Solution:
Step 1 The Auxiliary Equation for the given DE will be obtained by replacing $y'' \to \lambda^2$, $y' \to \lambda$ & $y \to 1$

$$\lambda^2 + 4\lambda + 4 = 0$$

The roots are then found as

$$\lambda_1 = \frac{-(4) + \sqrt{(4)^2 - 4(4)}}{2} \;\&\; \lambda_2 = \frac{-(4) - \sqrt{(4)^2 - 4(4)}}{2}$$

$$\lambda_1 = \frac{-4 + \sqrt{0}}{2} = -2 \;\&\; \lambda_2 = \frac{-4 - \sqrt{0}}{2} = -2$$

Step 2 The General Solution would be

$$y = (C_1 x + C_2)e^{-2x}$$

Step 3 The Particular Solution satisfying the initial conditions require

$$y(0) = [(C_1 x + C_2)e^{-2x}]_{x=0} = 3$$
$$C_2 = 3$$

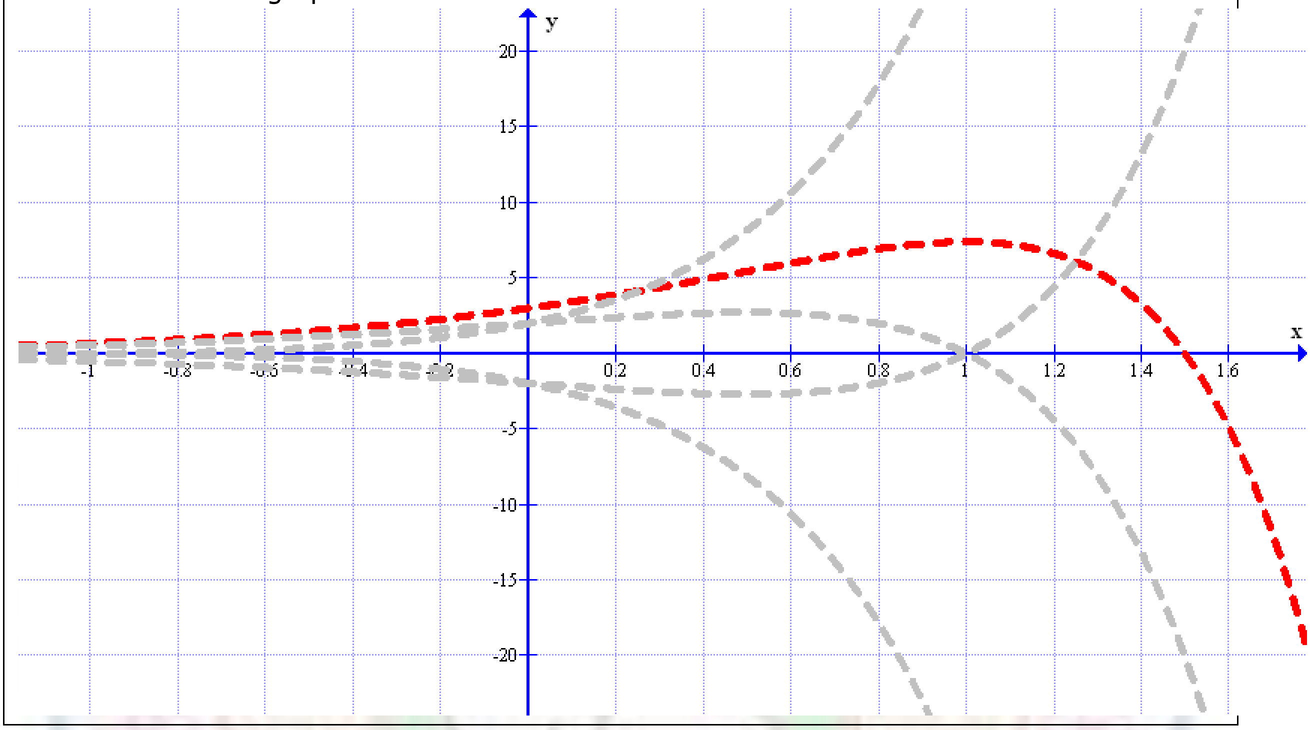
and

$$y'(0) = [C_1 e^{-2x} - 2(C_1 x + C_2)e^{-2x}]_{x=0} = 4$$
$$C_1 - 2C_2 = 4$$

On solving we get $C_1 = 10$ & $C_2 = 3$ so that the particular solution comes out to be

$$y = (10x + 3)e^{-2x}$$

Let's see it on the graph:



6.6.3 Case 3: Two complex but conjugate roots $\lambda_1 = \alpha + i\beta$ & $\lambda_2 = \alpha - i\beta$

If the two roots λ_1 & λ_2 are complex, we have $y_1 = e^{\lambda_1 x}$ & $y_1 = e^{\lambda_2 x}$ as the two solutions of the *DE* given by

$$y'' + ay' + by = 0$$

with

$$\lambda_1 = \frac{-a + i\sqrt{4b - a^2}}{2} \; \& \; \lambda_2 = \frac{-a - i\sqrt{4b - a^2}}{2}$$
$$\lambda_1 = \alpha + i\beta \; \& \; \lambda_2 = \alpha - i\beta$$

so that $\alpha = -a/2$ & $\beta = \sqrt{4b - a^2}/2$.

We find that

$$\frac{y_1(x)}{y_2(x)} = e^{(\lambda_1 - \lambda_2)x} = e^{i2\beta x}$$

is not a constant and so y_1 & y_2 are not linearly proportional to one another. Hence, they form a Basis of the solutions of the *DE*. The general solution would then be

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$
$$y = C_1 e^{(\alpha + i\beta)\,x} + C_2 e^{(\alpha - i\beta)\,x}$$
$$y = e^{\alpha x}\{C_1 e^{+i\beta x} + C_2 e^{-i\beta x}\}$$

Sometimes it is convenient to use the Euler's formula $e^{i\beta x} = \cos\beta x + i\sin\beta x$ to write

$$y = e^{\alpha x}\{C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)\}$$
$$y = e^{\alpha x}\{(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x\}$$
$$y = e^{\alpha x}\{A\cos\beta x + B\sin\beta x\}$$

and all such solution decay with increasing x if $\alpha < 0$. If $C_1 + C_2$ be any constant A and $i(C_1 + C_2)$ be any constant B then we find that $C_1 = (A - iB)/2$ and $C_1 = (A + iB)/2$ are themselves complex conjugate of each other.

Therefore, we eventually have

$$y = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$$

Putting

$$y_1(x) = e^{\alpha x}\cos\beta x \;\&\; y_2(x) = e^{\alpha x}\sin\beta x$$

Note again that $y_2/y_1 = \tan\beta x$ and they are not linearly proportional to one other. Hence, they form a Basis of the solutions of the *DE*. The general solution would then be

$$y = Ay_1(x) + By_2(x) = e^{\alpha x}\{A\cos\beta x + B\sin\beta x\}$$

where $\alpha = -a/2$ & $\beta = \sqrt{4b - a^2}/2$.

Example 6.6.3.1 Solve the initial value problem for the *Linear Homogeneous DE*

$$\boldsymbol{y'' - y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = \frac{7}{2}}$$

Solution:

Step 1 The Auxiliary Equation for the given *DE* will be obtained by replacing $y'' \to \lambda^2$, $y' \to \lambda$ & $y \to 1$

$$\lambda^2 - \lambda + 1 = 0$$

The roots are then found as

$$\lambda_1 = \frac{-(-1) + \sqrt{(-1)^2 - 4(1)}}{2} \;\&\; \lambda_2 = \frac{-(-1) - \sqrt{(-1)^2 - 4(1)}}{2}$$
$$\lambda_1 = \frac{1 + \sqrt{3}}{2} \;\&\; \lambda_2 = \frac{1 - i\sqrt{3}}{2}$$
$$\alpha = \frac{1}{2} \;\&\; \beta = \frac{\sqrt{3}}{2}$$

Step 2 The General Solution would be

$$y = e^{\alpha x}\{A\cos\beta x + B\sin\beta x\} = e^{\frac{1}{2}x}\left\{A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right\}$$

Step 3 The Particular Solution satisfying the initial conditions require

$$y(0) = \left[e^{\frac{1}{2}x}\left\{A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right\}\right]_{x=0} = 1$$
$$A = 1$$

and

$$y'(0) = \left[\frac{1}{2}e^{\frac{1}{2}x}\left\{A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right\} + \frac{\sqrt{3}}{2}e^{\frac{1}{2}x}\left\{-A\sin\frac{\sqrt{3}}{2}x + B\cos\frac{\sqrt{3}}{2}x\right\}\right]_{x=0} = \frac{7}{2}$$
$$\frac{A + B\sqrt{3}}{2} = \frac{7}{2}$$

On solving we get $A = 1$ & $B = 2\sqrt{3}$ so that the particular solution comes out to be

$$y = e^{\frac{1}{2}x}\left\{\cos\frac{\sqrt{3}}{2}x + 2\sqrt{3}\sin\frac{\sqrt{3}}{2}x\right\}$$

Let's see it on the graph:

Example 6.6.3.2 Solve the initial value problem the *Linear Homogeneous DE*

$$\boldsymbol{y'' + y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = \frac{7}{2}}$$

Solution:
Step 1 The Auxiliary Equation for the given DE will be obtained by replacing $y'' \to \lambda^2$, $y' \to \lambda$ & $y \to 1$

$$\lambda^2 + \lambda + 1 = 0$$

The roots are then found as

$$\lambda_1 = \frac{-(1) + \sqrt{(1)^2 - 4(1)(1)}}{2} \;\&\; \lambda_2 = \frac{-(1) - \sqrt{(1)^2 - 4(1)(1)}}{2}$$
$$\lambda_1 = \frac{-1 + \sqrt{3}}{2} \;\&\; \lambda_2 = \frac{-1 - i\sqrt{3}}{2}$$
$$\alpha = -\frac{1}{2} \;\&\; \beta = \frac{\sqrt{3}}{2}$$

Step 2 The General Solution would be

$$y = Ay_1(x) + By_2(x) = e^{-\frac{1}{2}x}\left\{A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right\}$$

Step 3 The Particular Solution satisfying the initial conditions require

$$y(0) = \left[e^{-\frac{1}{2}x}\left\{A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right\}\right]_{x=0} = 1$$
$$A = 1$$

and

$$y'(0) = \left[\left(-\frac{1}{2}\right)e^{-\frac{1}{2}x}\left\{A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right\} + \frac{\sqrt{3}}{2}e^{-\frac{1}{2}x}\left\{-A\sin\frac{\sqrt{3}}{2}x + B\cos\frac{\sqrt{3}}{2}x\right\}\right]_{x=0} = \frac{7}{2}$$

$$y'(0) = \left[\left(-\frac{1}{2}\right)\{A\} + \frac{\sqrt{3}}{2}\{B\}\right]_{x=0} = \frac{7}{2}$$

$$\frac{-A + B\sqrt{3}}{2} = \frac{7}{2}$$

$$-A + B\sqrt{3} = 7$$

On solving we get $A = 1$ & $B = 8/\sqrt{3}$ so that the particular solution comes out to be

$$y = e^{-\frac{1}{2}x}\left\{\cos\frac{\sqrt{3}}{2}x + \frac{8}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\right\}$$

Let's see it on the graph:

Summary

General Second Order 'Linear Homogeneous' Equations

- A general second order linear differential equation in $y(x)$ in the Normal Form or Standard Form is $y'' + p(x)y' + q(x)y = r(x)$ where
- A *nonlinear* equation will contain product terms of the type $y\,y'$, $y'y''$ etc.

Few properties of such DE

- *The solution $y(x) = 0$ is a* **particular solution** *of the Homogeneous Equation and is called the* ***trivial solution***.
- *If $y_{cf}(x)$ is a non-zero solution (known as the* ***Complementary Function****) of the Homogeneous Equation then $Ay_{cf}(x)$ is also a solution to the Homogeneous equation where A is an arbitrary constant.*
- *If the Homogeneous Equation admits* ***two Complementary Functions*** *$y_{1,cf}(x)$ & $y_{2,cf}(x)$ then $C_1y_{1,cf}(x) + C_2y_{2,cf}(x)$ is also a solution to the Homogeneous equation where C_1 & C_2 are some arbitrary constant.*
- *If $y_{1,cf}(x)$ & $y_{2,cf}(x)$ are* ***Linearly Independent solutions*** *($i.e.,C_1y_{1,cf}(x) + C_2y_{2,cf}(x) = 0$ on an interval I if and only if $C_1 = C_2 = 0$) of the Homogeneous Equation then every solution to the Homogeneous equation is given by the composite $y_{cf}(x) = C_1y_{1,cf}(x) + C_2y_{2,cf}(x)$ where C_1 & C_2 are arbitrary constants.*

Determination of Solution of Second Order Linear DE

- A solution of the second order (linear or nonlinear) *DE* on some open interval $a < x < b$ is function $y_{pi}(x)$ that has defined derivatives $y_{pi}'(x)$ and $y_{pi}''(x)$ and satisfies that *DE* for all x in that interval. In other words the *DE* becomes an identity if we replace the unknown function $y(x)$ & its derivatives by $y_{pi}(x)$ and its derivatives $y_{pi}'(x)$ & $y_{pi}''(x)$.

Theorem of Superposition of Solution to Second Order LDE

- *Theorem 1 : For the Second Order Linear Homogeneous DE*

$$y'' + p(x)y' + q(x)y = 0$$

any Linear combination $(C_1y_1(x) + C_2y_2(x))$ *of two solutions on an open interval* I *is again a solution on* I.

Initial value Solution of Second Order Linear DE

- The initial value problem now consists of two initial conditions $y(a) = b_0$ & $y'(a) = b_1$ which at some given point $x = a$ in the open interval have the solution and its derivative defined.

Basis General Solution and Particular Solution of Second Order Linear DE

- If the two solutions $y_{1,cf}(x)$ & $y_{2,cf}(x)$ are linearly independent i.e., we can say that on an interval I when the solutions are defined then $C_1y_1(x) + C_2y_2(x) = 0$ on I implies that $C_1, C_2 = 0$ then a pair $y_{1,cf}(x)$ & $y_{2,cf}(x)$ of linearly independent solutions form a **Basis** (or fundamental system) on the interval I.

Second Order Linear Homogeneous DE with Constant Coefficients

- A Second Order Linear Homogeneous *DE* with constant coefficients written as $y'' + ay' + by = 0$ has important applications in the field of physics and electrical circuits.
- If y were to be a non-zero solution of the *DE* then λ must satisfy the equation $\lambda^2 + a\lambda + b = 0$. This is called the *Auxiliary (Characteristics) Equation* and is quadratic in λ. Thus there will be two values of λ for which $y = e^{\lambda x}$ will be solution to the *DE*. If we denote the two roots as λ_1 & λ_2 then $e^{\lambda_1 x}$ & $e^{\lambda_2 x}$ will be two linearly independent solutions of the *DE*.

Depending on the sign of the discriminant we have the two roots λ_1 & λ_2 yielding solutions as in the following cases

- *Case 1: If* $a^2 - 4b > 0$ *then the two roots* λ_1 & λ_2 *are* ***real and distinct*** $(\lambda_1 \neq \lambda_2)$ *and* $e^{\lambda_1 x}$ & $e^{\lambda_2 x}$ *will be the two linearly independent solutions forming the Basis of the DE.*
- *Case 2: If* $a^2 - 4b = 0$ *then the two roots* λ_1 & λ_2 *are* ***real but equal*** $(\lambda_1 = \lambda_2 = \lambda)$ *and* $e^{\lambda x}$ & $xe^{\lambda x}$ *will be the two linearly independent solutions forming the Basis of the DE.*
- *Case 3: If* $a^2 - 4b < 0$ *then the two roots* λ_1 & λ_2 *are* ***complex but conjugate*** $(\lambda_1 = \alpha + i\beta$ & $\lambda_2 = \alpha - i\beta)$ *and* $e^{\alpha x} \cos \beta x$ & $e^{\alpha x} \sin \beta x$ *will be the two linearly independent solutions forming the Basis of the DE.*

Finally, for these cases the general solution will be accordingly:

- Case 1: Two real but distinct roots λ_1 & λ_2
  The general solution would then be $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$
- Case 2: Two real but equal roots $\lambda_1 = \lambda_2 = \lambda$
  The solution $y = (Ax + B)e^{\lambda x}$
- Case 3: Two complex but conjugate roots $\lambda_1 = \alpha + i\beta$ & $\lambda_2 = \alpha - i\beta$
  the solution $y = e^{\alpha x}\{A \cos \beta x + B \sin \beta x\}$

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