Discipline Course-I
Semester -I
Paper: Mathematical PhysicsI IA
Lesson: Second Order Linear Homogeneous Differential Equation (Continued)

Lesson Developer: Savinder Kaur
College/Department: SGTB Khalsa College, University of Delhi

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Learning Objectives

In this chapter the student will understand

- ***what is the Wronskian, a basis and how to check for linear independence and dependence of solutions of Second Order DE?***
- ***Fundamental Theorems on solutions of Second Order DE with constant coefficients.***
- ***Theorem of Existence of at least one solution for an initial value Second Order DE with constant coefficients.***
- ***Theorem of Uniqueness of at most one solution for an initial value Second Order DE with constant coefficients.***

Second Order Linear Homogeneous DE (continued)

7.1 The Wronskian

Consider the general second order linear homogeneous differential equation in $y(x)$ in the Normal Form or Standard Form

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x)$ and $q(x)$ are either constants or functions of x alone. By now we would have noticed that the solution to the above equation is made up of a Basis $y_1(x) \,\&\, y_2(x)$ which are linearly independent solutions to the DE.

Now let's recall that $y_1(x) \,\&\, y_2(x)$ will be *linearly independent* on an interval I if and only if

$$C_1y_1(x) + C_2y_2(x) = 0$$

on I implies that $C_1 = 0$ & $C_2 = 0$. Further, recall that $y_1(x) \,\&\, y_2(x)$ will be *linearly dependent* solutions on the interval I if we can express

$$y_1(x) = ay_2(x) \; or \; y_2(x) = by_1(x)$$

i.e., when $y_1 \,\&\, y_2$ are proportional.

A solution of the second order (linear or nonlinear) DE on some open interval $a < x < b$ is function $y_{pi}(x)$ that has defined derivatives $y_{pi}'(x)$ and $y_{pi}''(x)$ and satisfies that DE for all x in that interval. In other words the DE becomes an identity if we replace the unknown function $y(x)$ & its derivatives by $y_{pi}(x)$ and its derivatives $y_{pi}'(x) \,\&\, y_{pi}''(x)$. To understand the linear independence of the solutions we would like to now introduce an important quantity called, the Wronskian, W. For the two solutions above W is defined as

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

7.2 Theorem

***If the Second Order Linear Homogeneous* DE**

$$\boldsymbol{y'' + p(x)y' + q(x)y = 0} \qquad \textbf{(1)}$$

Has continuous coefficients $p(x) \,\&\, q(x)$ on an open interval I, then the two solutions $y_1(x) \,\&\, y_2(x)$ are Linearly Dependent on the open interval I;

(i) ***If and only if their Wronskian W is zero at some x_0 on I***

(ii) ***If the Wronskian $W = 0$ for $x = x_0$ then $W = 0$ at every x on I***

(iii) ***If and only if their Wronskian W is non- zero at some x_0 on I then the two solutions $y_1(x) \,\&\, y_2(x)$ are Linearly Independent on the open interval I***

Proof

(i) ***If and only if*** *their Wronskian W is zero at some x_0 on I*

If *we* consider $y_1(x)$ & $y_2(x)$ as two linearly dependent solutions of the *Homogeneous* DE on an interval I such that

$$y_1(x) = ky_2(x) \quad \forall\, x \in I$$

Their Wronskian will be

$$W(y_1, y_2) = W(ky_2, y_2) = \begin{vmatrix} ky_2 & y_2 \\ ky_2{}' & y_2{}' \end{vmatrix} = ky_2 \times y_2' - ky_2' \times y_2 = 0$$

for all $x \in I$. Therefore it is true for $x = x_0$.

Only if : *Conversely,* assume that for some $x = x_0$ on I, the Wronskian $W(y_1, y_2)_{x=x_0} = 0$ i.e.,

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) \end{vmatrix} = 0$$

Now for some non-zero values of C_1 & C_2 we can write

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

The reason for writing the above equation is the belief that the solutions $y_1(x)$ & $y_2(x)$ will finally come out to be as two linearly dependent solutions of the *Homogeneous* DE on an interval I. So we can write at $x = x_0$

$$C_1 y_1(x_0) + C_2 y_2(x_0) = 0$$

and

$$C_1 y_1{}'(x_0) + C_2 y_2{}'(x_0) = 0$$

Using these non-zero values of C_1 & C_2 we define a function

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Since $y_1(x)$ & $y_2(x)$ are solutions by ***Fundamental Theorem of Superposition, a linear combination of the two solutions must also be a solution***. Therefore, $y(x)$ is a solution with the property that at x_0

$$y(x_0) = 0$$
$$y'(x_0) = 0$$

However, we note that there is another function $Y(x) = 0 \;\; \forall\, x \in I$ (***the trivial solution***) which is a solution to the DE with the same property that

$$Y(x_0) = 0$$
$$Y'(x_0) = 0$$

Lastly, from the ***Uniqueness Theorem*** (discussed later in the chapter) a *particular* curve must pass through x_0, satisfying the above initial conditions (for both $Y(x)$ and $y(x)$

at x_0). Thus $Y(x)$ ***must be equal to*** $y(x)$ ***as two different curves cannot pass through the same point and have the same slope at that point without being identical***. So

$$y(x) = Y(x) = 0$$
$$C_1 y_1(x) + C_2 y_2(x) = 0$$
$$y_1(x) \propto y_2(x)$$

i.e. y_1 and y_2 are linearly dependent solutions. This theorem represents the Principle of Superposition and, should be noted, does not hold for *Non-Homogeneous / Non-Linear* DE.

(ii) *If the Wronskian $W = 0$ for $x = x_0$ then $W = 0$ at every x on I*

If $y_1(x)$ & $y_2(x)$ are two solutions of the second order *Homogeneous* DE (1) on an interval I containing a point x_0 then it follows

$$y_1''(x) = -p(x)y_1'(x) - q(x)y_1(x)$$
$$y_2''(x) = -p(x)y_2'(x) - q(x)y_2(x)$$

Their Wronskian will be

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \times y_2' - y_1' \times y_2$$

for all $x \in I$. And its derivative would be

$$W'(y_1, y_2) = y_1' \times y_2' + y_1 \times y_2'' - y_1'' \times y_2 - y_1' \times y_2'$$
$$W'(y_1, y_2) = y_1 \times y_2'' - y_1'' \times y_2$$
$$W'(y_1, y_2) = y_1 \times \{-py_2' - qy_2\} - \{-py_1' - qy_1\} \times y_2$$
$$W'(y_1, y_2) = -py_2'y_1 - qy_2y_1 + py_1'y_2 + qy_1y_2$$
$$W'(y_1, y_2) = py_1'y_2 - py_2'y_1$$
$$W'(y_1, y_2) = p(y_1'y_2 - y_2'y_1)$$
$$W'(y_1, y_2) = -pW(y_1, y_2)$$
$$W'(y_1, y_2) + pW(y_1, y_2) = 0$$

This is a first order linear DE which can be solved by separation of variable

$$\ln|W(x)| = -\int^x p(\xi)d\xi$$
$$W(x) = e^{-\int^x p(\xi)d\xi}$$

It would be true that at $x = x_0$

$$W(x_0) = e^{-\int^{x_0} p(\xi)d\xi}$$

Thus,

$$\frac{W(x)}{W(x_0)} = \frac{e^{-\int^x p(\xi)d\xi}}{e^{-\int^{x_0} p(\xi)d\xi}} = e^{-\int^x p(\xi)d\xi + \int^{x_0} p(\xi)d\xi} = e^{-\int_{x_0}^x p(\xi)d\xi}$$

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p(\xi)d\xi}$$

Since $p(\xi)$ is continuous on the interval I, thus the RHS is finite quantity so if $W(x_0) = 0$ on I then $W(x) = 0$ at every x on I.

(iii) ***If and only if*** *their Wronskian W is non- zero at some x_0 on I then the two solutions $y_1(x)$ & $y_2(x)$ are Linearly Independent on the open interval I*

Let $y_1(x)$ & $y_2(x)$ be two linearly independent solutions of the *Homogeneous* DE on an interval I containing a point x_0. Also let at the point x_0 the Wronskian of the two solutions $W(y_1, y_2)_{x=x_0} = 0$ so that

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_{x=x_0} = 0$$

We can find two non-zero constants C_1 & C_2 such that

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

If we define a function

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Then it is obvious that this function would satisfy the initial conditions

$$y(x_0) = C_1 y_1(x_0) + C_2 y_2(x_0) = 0$$

and

$$y'(x_0) = C_1 y_1'(x_0) + C_2 y_2'(x_0) = 0$$

The function $y(x)$ therefore is a solution to the DE satisfying the initial conditions. However, there exists a trivial solution $Y(x) = 0$ of the DE which satisfies the initial condition

$$Y(x_0) = 0$$

and

$$Y'(x_0) = 0$$

By Uniqueness Theorem we find

$$y(x) = Y(x)$$
$$C_1 y_1(x) + C_2 y_2(x) = 0$$
$$y_1(x) \propto y_2(x)$$

which means that $y_1(x)$ & $y_2(x)$ are linearly dependent. This however contradicts our initial assumption of $y_1(x)$ & $y_2(x)$ being linearly independent. ***Hence, if $y_1(x)$ & $y_2(x)$ are linearly independent then there is no point $x_0 \in I$ for which the Wronskian is zero $W(y_1, y_2)_{x=x_0} = 0$.***

Conversely, let $y_1(x)$ & $y_2(x)$ be two solutions of the DE such that at some point $x_0 \in I$ the Wronskian is non-zero $W(y_1, y_2)_{x=x_0} \neq 0$ and C_1 & C_2 be two non-zero constants such that

$$C_1 y_1(x) + C_2 y_2(x) = 0 \; \forall \; x \in I$$

From the second assumption we can write for some point $x_0 \in I$

$$C_1 y_1(x_0) + C_2 y_2(x_0) = 0$$

and

$$C_1 y_1'(x_0) + C_2 y_2'(x_0) = 0$$

We can write it in matrix form as

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

Since we have assumed that

$$W(y_1, y_2)_{x=x_0} = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

it implies that $C_1 \,\&\, C_2$ have to be zero. Thus, $y_1(x) \,\&\, y_2(x)$ as two linearly dependent solutions of the *Homogeneous* DE on an interval I since we cannot find any $C_1 \,\&\, C_2$ which are non-zero.

Examples of Wronskian:

Example 7.2.1 Check that the two solutions of the *Linear Homogeneous* DE are independent or not

$$\boldsymbol{y'' + k^2 y = 0}$$

Solution:
Step 1 The Auxiliary Equation for the given DE will be obtained by replacing $y'' \to \lambda^2$, $y' \to \lambda$ & $y \to 1$

$$\lambda^2 + k^2 = 0$$

The roots are then found as

$$\lambda^2 = -k^2$$
$$\lambda = ik \,\&\, \lambda = -ik$$
$$\alpha = 0 \,\&\, \beta = k$$

Step 2 The two solutions would be

$$y_1 = e^{0\times x}\cos kx = \cos kx$$
$$y_2 = e^{0\times x}\sin kx = \sin kx$$

Step 3 We now determine the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx \end{vmatrix}$$
$$W(y_1, y_2) = k\cos^2 kx + k\sin^2 kx = k(\cos^2 kx + \sin^2 kx)$$
$$W(y_1, y_2) = k$$

and the two solutions $\cos kx$ & $\sin kx$ will be linearly independent if and only if $k \neq 0$.

7.3 Fundamental Theorems

Consider a general *Homogeneous* Second Order Linear DE

$$L(y) = y'' + a_1 y' + a_0 y = 0$$

From its Characteristics Equation

$$\lambda^2 + a_1\lambda + a_0 = 0$$

we have been seeing that it has two linearly independent solutions y_1 & y_2. We now write it formally as the following theorem:

7.3.1 The Theorem

Suppose that a_1 & a_0 are two arbitrary constants and there is an equation of the form

$$\boldsymbol{L(y) = y'' + a_1 y' + a_0 y = 0}$$

If λ_1 & λ_2 are two distinct roots of the characteristics polynomial

$$\boldsymbol{p(\lambda) = \lambda^2 + a_1\lambda + a_0 = 0}$$

then the functions $y_1 = e^{\lambda_1 x}$ & $y_2 = e^{\lambda_2 x}$ are the solutions of the equation $L(y) = 0$. If, however, $\lambda_1 = \lambda_2 = \lambda$ is a repeated root of the polynomial $p(\lambda) = 0$ then the functions $y_1 = e^{\lambda x}$ & $y_2 = xe^{\lambda x}$ are the solutions of the equation $L(y) = 0$.

Example 7.3.1.1 Find the independent solutions for the *Linear Homogeneous* DE

$$\boldsymbol{y'' + 9y = 0}$$

Solution:

Step 1 The Auxiliary Equation for the given DE will be obtained by replacing $y'' \to \lambda^2$, $y' \to \lambda$ & $y \to 1$

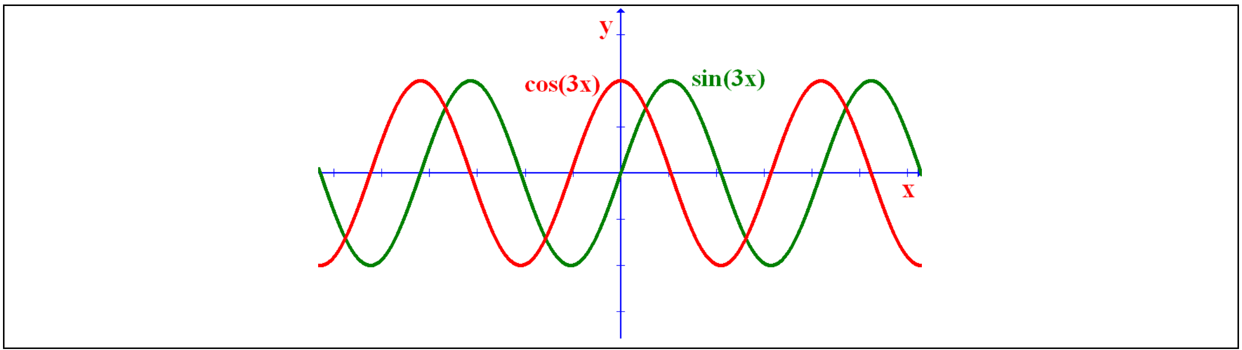
$$\lambda^2 + 9 = 0$$

The roots are then found as

$$\lambda^2 = -9$$
$$\lambda = 3i \; \& \; \lambda = -3i$$
$$\alpha = 0 \; \& \; \beta = 3$$

Step 2 The two independent solutions would be

$$y_1 = e^{0\times x}\cos 3x = \cos 3x$$
$$y_1 = e^{0\times x}\sin 3x = \sin 3x$$



We have also learnt that all general solutions y must be a linear combination of the two independent solutions y_1 & y_2. Now consider a general *Homogeneous* Second Order Linear DE

$$L(y) = y'' + a_1 y' + a_0 y = 0$$

with some initial or boundary condition on the value of the solution y and its derivative y' at some point x_0. The general solution would be

$$y = C_1 y_1 + C_2 y_2$$

of which some combination of C_1 & C_2 would satisfy the initial conditions. It's natural to ask whether such solution exists and whether they would be unique?

7.3.2 Theorem of Existence

There exists a solution y of the initial value problem

$$\boldsymbol{L(y) = y'' + a_1 y' + a_0 y = 0}$$

on $-\infty < x < \infty$ for any real x_0 and constants $y(x_0) = \alpha$ and $y'(x_0) = \beta$.

Proof
We know that for an equation $L(y) = 0$ we can write the characteristic polynomial equation $p(\lambda) = 0$. The roots of the polynomial will then determine the two linearly independent solutions of the DE. We have two ways to write the solution $y_1(x)$ & $y_2(x)$ corresponding to these roots
(i) If λ_1 & λ_2 are distinct then $y_1(x) = e^{\lambda_1 x}$ & $y_2(x) = e^{\lambda_2 x}$
(ii) If λ_1 & λ_2 are equal (λ) then $y_1(x) = e^{\lambda x}$ & $y_2(x) = xe^{\lambda x}$
We shall now show that there exist constants C_1 & C_2 such that

$$y = C_1 y_1 + C_2 y_2$$

satisfies $y(x_0) = \alpha$ and $y'(x_0) = \beta$ where x_0 is some real number and α, β are given constants.
Following these conditions we have at x_0

$$C_1 y_1(x_0) + C_2 y_2(x_0) = \alpha$$
$$C_1 y_1'(x_0) + C_2 y_2'(x_0) = \beta$$

These equations will have *non-zero* C_1, C_2 if the determinant

$$\Delta = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

We check this for the two cases

(i) When the roots are distinct

$$\Delta = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0)$$
$$\Delta = e^{\lambda_1 x_0} \times \lambda_2 e^{\lambda_2 x_0} - e^{\lambda_2 x_0} \times \lambda_1 e^{\lambda_1 x_0} = (\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)x_0}$$

This is not zero as $\lambda_2 \neq \lambda_1$ and $e^{(\lambda_1+\lambda_2)x_0} \neq 0$. Thus,

$$\Delta \neq 0$$

(ii) When the roots are equal

$$\Delta == \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0)$$
$$\Delta = \left[e^{\lambda x} \times \frac{d}{dx}\left(xe^{\lambda x}\right) - \left(xe^{\lambda x}\right) \times \frac{d}{dx}\left(e^{\lambda x}\right) \right]\bigg|_{x=x_0}$$
$$\Delta = e^{\lambda x_0} \times \left(e^{\lambda x_0} + x\lambda e^{\lambda x_0}\right)\Big|_{x=x_0} - \left(x_0 e^{\lambda x_0}\right) \times \lambda e^{\lambda x_0}$$
$$\Delta = (1 + x_0\lambda - x_0\lambda)e^{2\lambda x_0} = e^{2\lambda x_0}$$

This is also not zero as $e^{2\lambda x_0} \neq 0$. Thus,

$$\Delta \neq 0$$

Since in both the cases the determinants are *non-zero* for any choice of $x_0, \alpha \,\&\, \beta$, we will always find some *non-zero* $C_1 \,\&\, C_2$. This guarantees existence of a solution

$$y = C_1 y_1 + C_2 y_2$$

From the above theorem, we have come to know that a general Homogeneous Second Order Linear DE

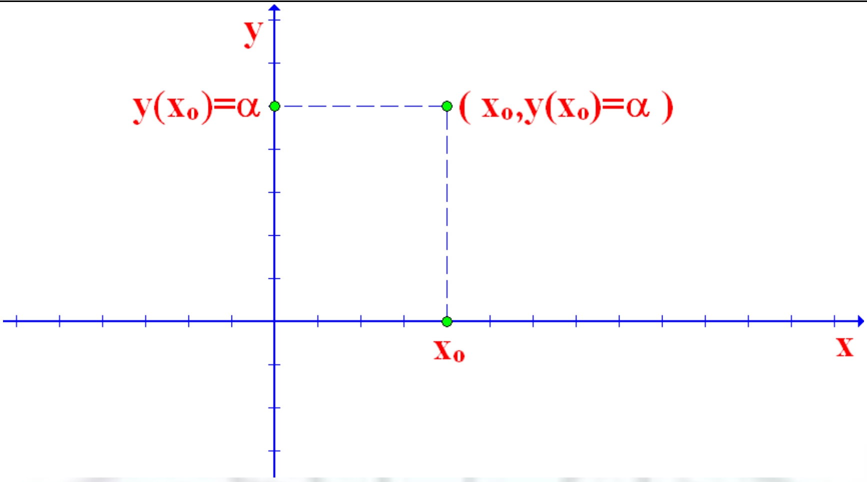
$$\boldsymbol{L(y) = y'' + a_1 y' + a_0 y = 0}$$

with some initial or boundary condition on the value of the solution y and its derivative y' at some point x_0 must have a solution

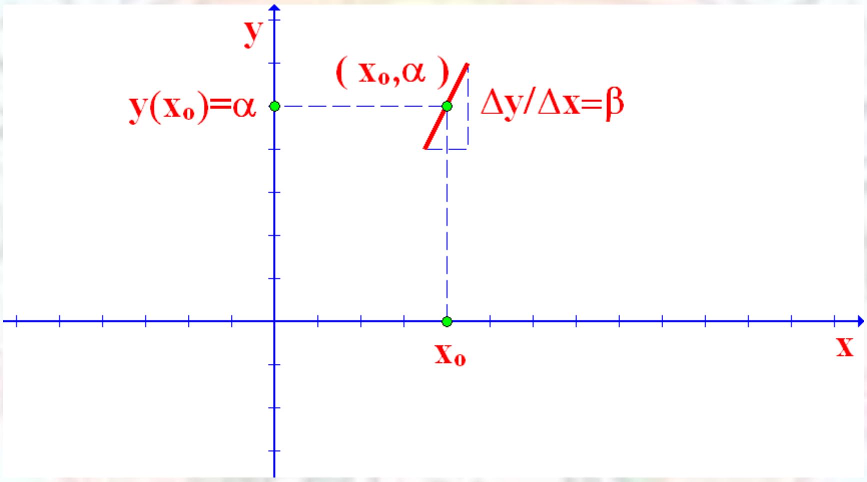
$$\boldsymbol{y = C_1 y_1 + C_2 y_2}$$

Understanding for Students

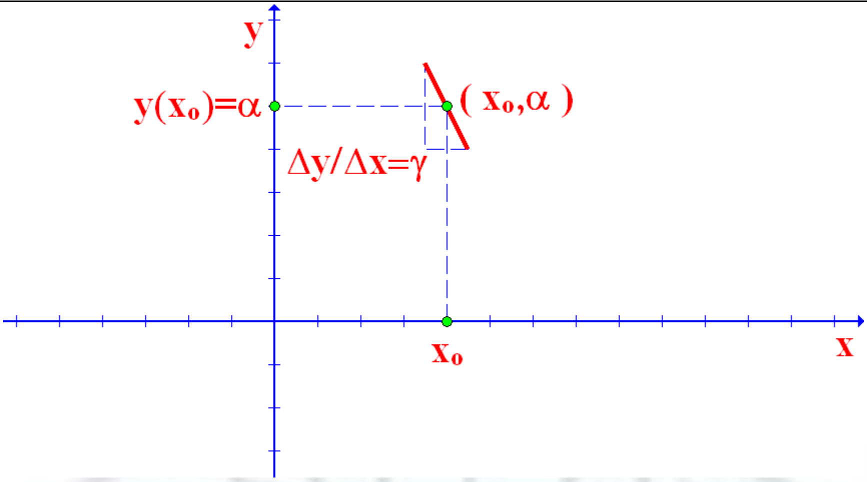
If we plot the solution y vs x, then the initial condition $y(x_0) = \alpha$ represents the point on the $y - x$ plane as shown in the figure



Now if $y'(x_0) = \beta$, then this defines the rate of change of y at the point x_0 to be equal to β. This means that at the point (x_0, α) there passes a curve with a slope of β.

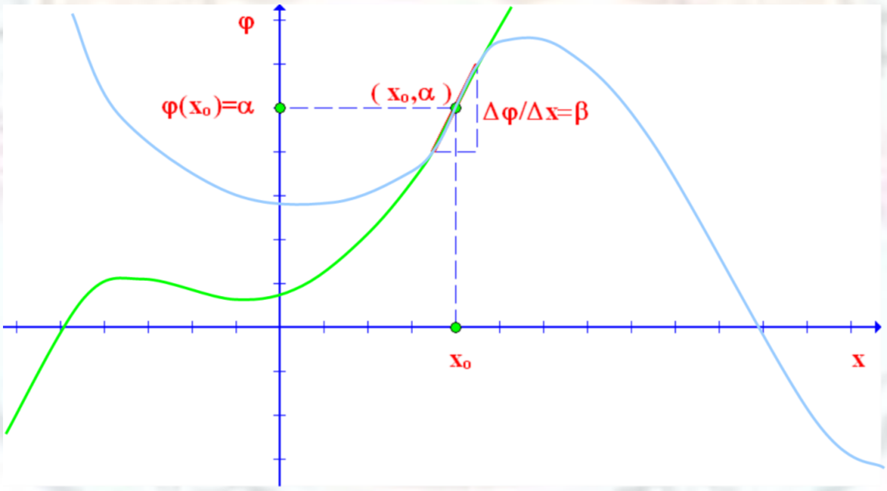


We may now ask the question in accordance to the existence theorem, will there be any other curve passing through the same point (x_0, α)? Now suppose if we also have $y'(x_0) = \gamma$, then we shall have another curve passing through the same point (x_0, α) but having γ as the value of the slope. This means that we have found atleast two curves passing through the same point (x_0, α) but with different slopes. These two curves represent **different initial conditions** $y(x_0) = \alpha, y'(x_0) = \beta$ and $y(x_0) = \alpha, y'(x_0) = \gamma$.



We find that the *Existence Theorem* allows *at least one solution* for each of these cases with different initial conditions.

Now the next question could be: will there be two ***different curves*** passing through the ***same point*** having ***same slope*** at that point? The figure below shows such a hypothetical situation (*the blue and the green curves pass through the same point and have the sae same slope at that point*) and the next theorem explicitly tells us that only one curve will pass through the point (x_0,α) with the same initial conditions.



From the uniqueness theorem we see later that only one type of curve can pass the point (x_0, α) with fixed initial conditions.

Example 7.3.2.1 Find the initial value $y(0) = 0$ & $y'(0) = 3$ solution for the *Linear Homogeneous* DE

$$y'' + 9y = 0$$

Solution:
Step 1 The Auxiliary Equation for the given DE will be obtained by replacing $y'' \to \lambda^2$, $y' \to \lambda$ & $y \to 1$

$$\lambda^2 + 9 = 0$$

The roots are then found as

$$\lambda^2 = -9$$
$$\lambda = 3i \,\&\, \lambda = -3i$$
$$\alpha = 0 \,\&\, \beta = 3$$

Step 2 The two independent solutions would be

$$y_1 = e^{0\times x}\cos 3x = \cos 3x$$
$$y_2 = e^{0\times x}\sin 3x = \sin 3x$$

Step 3 We now write the general solution

$$y(x) = C_1\cos 3x + C_2 \sin 3x$$

Step 4 Since $y(0) = 0 = C_1\cos 0 + C_2\sin 0 = C_1$ and $y'(0) = 3 = -3C_1\sin 0 + 3C_2\cos 0 = 3C_2$, we find $C_1 = 0 \,\&\, C_2 = 1$. The solution is thus

$$y(x) = \sin 3x \quad \forall \quad x \in I$$

Example 7.3.2.2 Find the initial value $y(0) = 0 \,\&\, y'(0) = 1$ solution for the *Linear Homogeneous* DE

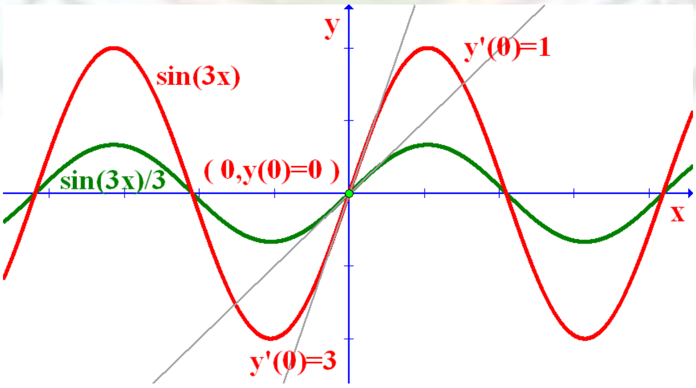
$$y'' + 9y = 0$$

Solution: Since the Steps 1, 2 & 3 are same as above, we expect a general solution of the form

$$y(x) = C_1\cos 3x + C_2 \sin 3x$$

Step 4 Since $y(0) = 0 = C_1\cos 0 + C_2\sin 0 = C_1$ and $y'(0) = 1 = -3C_1\sin 0 + 3C_2\cos 0 = 3C_2$, we find $C_1 = 0 \,\&\, C_2 = 1/3$. The solution is thus

$$y(x) = \frac{1}{3}\sin 3x \quad \forall \quad x \in I$$



In the figure above, the **red curve** represents the solution satisfying the initial condition $y(0) = 0 \,\&\, y'(0) = 3$ and the **green curve** represents the solution satisfying the initial condition $y(0) = 0 \,\&\, y'(0) = 1$.

We now answer the question that how many of solutions can be there which satisfies the same initial conditions? That is to say, is the combination of constants C_1, C_2 unique for a given initial condition?

7.3.3 Theorem of Uniqueness

On any interval I, a linear DE

$$\boldsymbol{L(y) = y'' + a_1 y' + a_0 y = 0}$$

on $-\infty < x < \infty$ ***with the initial conditions*** $y = \alpha$ **and** $y' = \beta$ **for any real** x_0 **has at most one solution**

$$\boldsymbol{y = C_1 y_1 + C_2 y_2}$$

Proof

Let us suppose that the DE

$$L(y) = 0$$

has two different solutions $Y_1(x)$ & $Y_2(x)$ (as linear combination of the ***independent solutions*** y_1 & y_2)

$$Y_1 = C_1 y_1 + C_2 y_2$$
$$Y_2 = C_3 y_1 + C_4 y_2$$

The two different solutions Y_1 & Y_2 satisfy the same initial conditions

$$Y_1(x_0) = \alpha \;\&\; Y_1{}'(x_0) = \beta$$
$$Y_2(x_0) = \alpha \;\&\; Y_2{}'(x_0) = \beta$$

Since they are solutions to the DE, they satisfy

$$L(Y_1) = 0 \;\;\&\;\; L(Y_2) = 0$$

Let us now define another function $Y(x) = Y_1(x) - Y_2(x)$, we then find that

$$L(Y) = L(Y_1 - Y_2) = L(Y_1) - L(Y_2) = 0 - 0$$
$$L(Y) = 0$$

Therefore the new function $Y(x)$ must also be a solution of the DE.

Further if we find the value of the new function and its derivative at the point x_0

$$Y(x_0) = Y_1(x_0) - Y_2(x_0) = \alpha - \alpha = 0$$
$$Y'(x_0) = Y_1{}'(x_0) - Y_2{}'(x_0) = \beta - \beta = 0$$

We would like to state here that any solution $Y(x)$ **of the DE** $L(y) = 0$ **having the initial conditions** $Y(x_0) = 0$ & $Y'(x_0) = 0$ **must be a *trivial solution* of the DE** $L(y) = 0$**.** Thus, being the trivial solution, $Y(x) = 0$ for all $x \in I$ and so it's first derivative $Y'(x) = 0$.

The above **conclusion** is drawn from the following arguments:
For any $Y(x)$ being a solution of the DE $L(y) = 0 \;\; \forall\, x \in I$ containing any point x_0 in the interval I, the modulus of the solution & the rate of growth of the DE $L(y) = y'' + a_1 y' + a_0 y = 0$ can be defined by

$$\|Y(x)\| = \sqrt{|Y(x)|^2 + |Y'(x)|^2} \quad \& \quad k = 1 + a_1 + a_0$$

and thus for all x in I it can be shown that

$$\|Y(x_0)\| e^{-k|x-x_0|} \le \|Y(x)\| \le \|Y(x_0)\|\, e^{k|x-x_0|}$$

Since $Y(x_0) = 0$ and $Y'(x_0) = 0$ we find $\|Y(x_0)\| = \sqrt{|Y(x_0)|^2 + |Y'(x_0)|^2} = 0$ and thus

$$\|Y(x_0)\| e^{-k|x-x_0|} \le \|Y(x)\| \le \|Y(x_0)\|\, e^{k|x-x_0|} \;\; \rightarrow \;\; 0 \le \|Y(x)\| \le 0$$
$$\|Y(x)\| = 0$$

This implies that for all values of $x \in I$

$$\|Y(x)\| = \sqrt{|Y(x)|^2 + |Y'(x)|^2} = 0$$
$$|Y(x)|^2 + |Y'(x)|^2 = 0$$

which will be satisfied only if we separately have

$$Y(x) = 0 \;\; \& \;\; Y'(x) \;\; \forall \;\; x \in I$$

This then means that for all $x \in I$

$$Y(x) = Y_1(x) - Y_2(x) = 0 \;\; \rightarrow \;\; Y_1(x) = Y_2(x)$$
$$Y'(x) = Y_1'(x) - Y_2'(x) = 0 \;\; \rightarrow \;\; Y_1'(x) = Y_2'(x)$$

Thus, for $Y_1(x) = Y_2(x)$ and we get

$$C_1 y_1 + C_2 y_2 = C_3 y_1 + C_4 y_2 \;\; \rightarrow \;\; (C_1 - C_3) y_1 = (C_4 - C_2) y_2$$

Since, the functions $y_1 \,\&\, y_2$ are independent solutions of the DE $L(y) = 0$, we find $C_1 = C_3 \;\&\; C_4 = C_2$ and so

$$Y_1(x) = Y_2(x)$$

Therefore, the assumption that $Y_1(x) \,\&\, Y_2(x)$ are two different solutions of the DE $L(y) = 0$ satisfying **the same initial condition** ($y = \alpha$ and $y' = \beta$ for any real x_0) does not hold and **we conclude that the solution must be Unique**.

Example 7.3.3.1 For the initial conditions $y'(\alpha) = 0 \,\&\, y(\alpha) = 0$ find the solution for the *Linear Homogeneous* DE

$$\boldsymbol{y'' + 9y = 0}$$

Solution:
Step 1 The Auxiliary Equation for the given DE will be obtained by replacing $y'' \to \lambda^2$, $y' \to \lambda$ & $y \to 1$

$$\lambda^2 + 9 = 0$$

The roots are then found as

$$\lambda^2 = -9$$
$$\lambda = 0 + 3i \;\&\; \lambda = 0 - 3i$$

Step 2 The two independent solutions would be

$$y_1 = e^{0\times x}\cos 3x = \cos 3x$$
$$y_2 = e^{0\times x}\sin 3x = \sin 3x$$

Step 3 We now write the general solution as a linear combination of the above independent solutions as

$$y(x) = C_1 \cos 3x + C_2 \sin 3x$$

Step 4 Using the initial conditions

$$y(\alpha) = 0 = C_1 \cos 3\alpha + C_2 \sin 3\alpha$$
$$tan(3\alpha) = -C_1/C_2$$

and

$$y'(\alpha) = 0 = -3C_1 \sin \alpha + 3C_2 \cos \alpha$$
$$tan(3\alpha) = C_2/C_1$$

we find

$$-\frac{C_1}{C_2} = \frac{C_2}{C_1}$$

or

$$C_1^2 + C_2^2 = 0$$

Which is possible only if $C_1 = 0$ and $C_2 = 0$. The **trivial** solution is thus

$$y(x) = 0 \quad \forall \quad x \in I$$

Example 7.3.3.2 Find for the initial values $y'(x_0) = 0$ & $y(x_0) = 0$ the solution for the *Linear Homogeneous* DE

$$\boldsymbol{y'' + \beta y' + \alpha y = 0}$$

Solution:
Step 1 The Auxiliary Equation for the given DE will be obtained by replacing $y'' \to \lambda^2$, $y' \to \lambda$ & $y \to 1$

$$\lambda^2 + \beta\lambda' + \alpha\lambda = 0$$

The general solution y written in terms of linearly independent solutions $y_1(x)$ & $y_2(x)$

(obtained corresponding to values of λ_1 & λ_2) is

$$y = C_1 y_1(x) + C_2 y_2(x)$$

Step 2 The initial conditions demand

$$y(x_0) = 0 = C_1 y_1(x_0) + C_2 y_2(x_0)$$
$$y'(x_0) = 0 = C_1 {y_1}'(x_0) + C_2 {y_2}'(x_0)$$

Step 3 We can write these two conditions in the following form

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ {y_1}'(x_0) & {y_2}'(x_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

Step 4 Since

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ {y_1}'(x_0) & {y_2}'(x_0) \end{pmatrix} \neq 0$$

as y_1 and y_2 are linearly independent solutions and so their Wronskian cannot be zero at x_0 (and hence also not at any value of x). The solution is possible when

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$
$$y(x) = 0 \quad \forall \quad x \in I$$

Summary

The Wronskian

- For a general second order linear homogeneous differential equation in $y(x)$ in the Normal Form or Standard Form $y'' + p(x)y' + q(x)y = 0$ where $p(x)$ and $q(x)$ are either constants or functions of x alone, we have noticed that the solution to the above equation is made up of a Basis $y_1(x)$ & $y_2(x)$ which are linearly independent solutions to the DE.
- The solutions $y_1(x)$ & $y_2(x)$ will be *linearly independent* on an interval I if and only if $C_1 y_1(x) + C_2 y_2(x) = 0$ on I implies that $C_1 = 0$ & $C_2 = 0$. Further, recall that $y_1(x)$ & $y_2(x)$ will be *linearly dependent* solutions on the interval I if we can express $y_1(x) = a y_2(x)$ *or* $y_2(x) = b y_1(x)$ i.e., when y_1 & y_2 are proportional.
- the Wronskian, W, for the two solutions above is defined as

$$W = \begin{vmatrix} y_1 & y_2 \\ {y_1}' & {y_2}' \end{vmatrix} = y_1 {y_2}' - y_2 {y_1}'$$

Theorem

- *If the Second Order Linear Homogeneous* DE $y'' + p(x)y' + q(x)y = 0$ *Has continuous coefficients* $p(x)$ & $q(x)$ *on an open interval* I*, then the two solutions* $y_1(x)$ & $y_2(x)$ *are* ***Linearly Dependent*** *on the open interval* I;
  (i) ***If*** *and only if their Wronskian* W *is zero* ***at some*** $\boldsymbol{x_0}$ *on* I
  (ii) ***If*** *the Wronskian* $W = 0$ *for* $x = x_0$ *then* $W = 0$ ***at every*** $\boldsymbol{x}$ *on* I
  (iii) ***If and only if*** *their Wronskian* $\boldsymbol{W}$ ***is non- zero*** *at some* x_0 *on* I *then the two*

solutions $y_1(x)$ & $y_2(x)$ *are* ***Linearly Independent*** *on the open interval* I

Fundamental Theorems

The Theorem

- *Suppose that* a_1 & a_0 *are two arbitrary constants and there is an equation of the form* $L(y) = y'' + a_1 y' + a_0 y = 0$ *. If* λ_1 & λ_2 *are two distinct roots of the characteristics polynomial* $p(\lambda) = \lambda^2 + a_1\lambda + a_0 = 0$ *then the functions* $y_1 = e^{\lambda_1 x}$ & $y_2 = e^{\lambda_2 x}$ *are the solutions of the equation* $L(y) = 0$*. If, however,* $\lambda_1 = \lambda_2 = \lambda$ *is a repeated root of the polynomial* $p(\lambda) = 0$ *then the functions* $y_1 = e^{\lambda x}$ & $y_2 = xe^{\lambda x}$ *are the solutions of the equation* $L(y) = 0$*.*

Theorem of Existence

- *There exists a solution* $\mathbf{y}$ *of the initial value problem* $L(y) = y'' + a_1 y' + a_0 y = 0$ *on* $-\infty < x < \infty$ *for any real* x_0 *and constants* $y(x_0) = \alpha$ and $y'(x_0) = \beta$.

Theorem of Uniqueness

- *On any interval I, a linear DE* $L(y) = y'' + a_1 y' + a_0 y = 0$ *on* $-\infty < x < \infty$ *with the initial conditions* $y = \alpha$ and $y' = \beta$ for any real x_0 has at most one solution $y = C_1 y_1 + C_2 y_2$

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