Discipline Course-I
Semester -I
Paper: Mathematical PhysicsI IA
Lesson: The D Operator & the Non-Homogeneous Equation

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Learning Objective

The student evolves further to calculate the solution of the Non-Homogeneous DE by finding the

- **Particular Integral for Special Forms of the Function $f(x)$ in the Non-Homogeneous DE**
- **rules to determine PI in shorter steps and learns the D-Operator**
  - **When function $f(x)$ is of the form e^{ax}**
  - **When function $f(x)$ is of the form $\sin ax$ *or* $\cos ax$**
  - **When function $f(x)$ is of the form x^m, m being a positive integer**
  - **When function $f(x)$ is of the form $e^{ax}V(x)$**

The D Operator & the Non-Homogeneous Equation

9.1 Particular Integral of Special Forms of the Function f(x)

As the previous two examples may have suggested finding PI could be very difficult involving tedious integrations. However, there are certain special forms of the function $f(x)$ which admits rules for finding PI in shorter steps. We would explore such functions and show our confidence in the rules developed;

9.2 *When function $f(x)$ is of the form e^{ax}*

If $f(x) = e^{ax}$ then we can see that

$$De^{ax} = ae^{ax}$$
$$D^2e^{ax} = D(De^{ax}) = D(ae^{ax}) = a^2e^{ax}$$

and so on

$$D^n e^{ax} = a^n e^{ax}$$

So if

$$L(D) = a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$

then

$$L(D)e^{ax} = (a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)e^{ax}$$
$$L(D)e^{ax} = (a_n a^n + a_{n-1}a^{n-1} + \cdots + a_1 a + a_0)e^{ax}$$
$$L(D)e^{ax} = L(a)e^{ax}$$

Thus, operating on both sides by the "*inverse*" operator $\frac{1}{L(D)}$ we find that

$$\frac{1}{L(D)}L(D)e^{ax} = \frac{1}{L(D)}L(a)e^{ax}$$
$$e^{ax} = L(a)\frac{1}{L(D)}e^{ax}$$

Now if $L(a) \neq 0$ this can be interpreted as

$$\frac{1}{L(D)}e^{ax} = \frac{e^{ax}}{L(a)}$$

This beautiful result then states a rule that an n^{th} order Non-Homogeneous Linear DE with Constant coefficients

$$L(D)y = Ae^{ax}$$

has the PI

$$y = A\frac{e^{ax}}{L(a)}$$

which needs no integration to be performed.

There may arise a situation where $L(a) = 0$. This would then imply "a" to be an r^{th} *order* *root* of the n^{th} *order Non-Homogeneous Linear DE with Constant coefficients so that*

$$L(D) = (D-a)^r \varphi(D).$$

The DE can then be written as

$$\varphi(D)(D-a)^r y = Ae^{ax}$$

Operating on both sides by the "*inverse*" operator $\frac{1}{\varphi(D)}$ we find

$$\frac{1}{\varphi(D)}\varphi(D)(D-a)^r y = \frac{1}{\varphi(D)}Ae^{ax}$$

$$(D-a)^r y = A\frac{e^{ax}}{\varphi(a)}$$

From our previously learnt technique this yields

$$y = \frac{A}{\varphi(a)}\frac{x^r}{r!}e^{ax}$$

Example 9.2.1 Solve the equation

$$\boldsymbol{y'' + y' + y = e^{-x}}$$

Solution:

Step 1 The *DE* will be written with the D operator by replacing $y'' \to D^2y$ & $y' \to Dy$

$$(D^2 + D + 1)y = e^{-x}$$

$$L(D)y = e^{-x}$$

Step 2 The *Auxiliary Equation* for the corresponding homogeneous *DE*

$$L(D)y = 0$$

will be obtained by writing

$$L(\lambda) = 0$$

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda_1 = \frac{-1+\sqrt{1^2-4}}{2} \ \& \ \lambda_2 = \frac{-1-\sqrt{1^2-4}}{2}$$

$$\lambda_1 = \frac{-1+\sqrt{-3}}{2} \ \& \ \lambda_2 = \frac{-1-\sqrt{-3}}{2}$$

The roots are then found as

$$\lambda_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \ \& \ \lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

The CF would be $C_1 e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x} + C_2 e^{\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x}$ which can be represented as

$$CF = e^{-\frac{x}{2}}\left\{C_1\cos\left(\frac{\sqrt{3}}{2}x\right) + C_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$$

Step 3 The PI would now be obtained as

$$PI = \frac{1}{L(D)} e^{-x}$$

Since $f(x) = e^{-x}$ is an exponential function we will use the rule $\frac{1}{L(D)} e^{ax} = \frac{e^{ax}}{L(a)}$ to find the PI

$$PI = \frac{1}{L(D)} e^{-x} = \frac{e^{-x}}{L(-1)}$$
$$PI = \frac{e^{-x}}{\{(-1)^2 + (-1) + 1\}}$$
$$PI = \frac{e^{-x}}{\{1 + (-1) + 1\}} = e^{-x}$$

Step 4 The *General Solution* would therefore be

$$y = CF + PI = e^{-\frac{x}{2}} \left\{ C_1 \cos\left(\frac{\sqrt{3}}{2} x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2} x\right) \right\} + e^{-x}$$

Example 9.2.2 Solve the equation

$$\boldsymbol{y'' - 4y' + 4y = e^x}$$

Solution:
Step 1 The *DE* will be written with the D operator by replacing $y'' \to D^2 y$ & $y' \to Dy$

$$(D^2 - 4D + 4)y = e^x$$
$$L(D)y = e^x$$

Step 2 The *Auxiliary Equation* for the corresponding homogeneous *DE*

$$L(D)y = 0$$

will be obtained by writing

$$L(\lambda) = 0$$
$$\lambda^2 - 4\lambda + 4 = 0$$
$$\lambda_1 = \frac{-(-4) + \sqrt{(-4)^2 - 4 \times 4}}{2} \ \& \ \lambda_2 = \frac{-(-4) - \sqrt{(-4)^2 - 4 \times 4}}{2}$$
$$\lambda_1 = 2 \ \& \ \lambda_2 = 2$$

The roots are then found to a double root

$$\lambda_1 = \lambda_2 = 2$$

The CF would be

$$CF = (C_1 x + C_2)e^{2x}$$

Step 3 The PI would now be obtained as

$$PI = \frac{1}{L(D)} e^x$$

Since $f(x) = e^{-x}$ is an exponential function we will use the rule $\frac{1}{L(D)}e^{ax} = \frac{e^{ax}}{L(a)}$ to find the PI

$$PI = \frac{1}{L(D)}e^{x} = \frac{e^{x}}{L(1)}$$
$$PI = \frac{e^{-x}}{\{1^2 - 4(1) + 4\}}$$

$$PI = e^{x}$$

Step 4 The *General Solution* would therefore be

$$y = CF + PI = (C_1 x + C_2)e^{2x} + e^{x}$$

9.3 *When function $f(x)$ is of the form $\sin ax$ or $\cos ax$*

If $f(x) = \sin(ax + \theta)$ then we can see that

$$D\sin(ax+\theta) = a\cos(ax+\theta)$$
$$D^2\sin(ax+\theta) = D(D\sin(ax+\theta)) = D(a\cos(ax+\theta)) = -a^2\sin(ax+\theta)$$
$$D^3\sin(ax+\theta) = -a^3\cos(ax+\theta)$$
$$D^4\sin(ax+\theta) = a^4\cos(ax+\theta) = (-a^2)^2\cos(ax+\theta)$$

and so on

$$(D^2)^n\sin(ax+\theta) = (-a^2)^n\sin(ax+\theta)$$

So if

$$L(D) = a_nD^{2n} + \cdots + a_2D^4 + a_1D^2 + a_0$$

contains only even powers of the operator D then it can be seen as polynomial φ in D^2 of power n so that

$$\varphi(D^2)\sin(ax+\theta) = \{a_n(D^2)^n + \cdots + a_2(D^2)^2 + a_1(D^2) + a_0\}\sin(ax+\theta)$$
$$\varphi(D^2)\sin(ax+\theta) = \{a_n(-a^2)^n + \cdots + a_2(-a^2)^2 + a_1(-a^2) + a_0\}\sin(ax+\theta)$$
$$\varphi(D^2)\sin(ax+\theta) = \varphi(-a^2)\sin(ax+\theta)$$

Thus, operating on both sides by the "*inverse"* operator $\frac{1}{\varphi(D^2)}$ we find that

$$\frac{1}{\varphi(D^2)}\varphi(D^2)\sin(ax+\theta) = \frac{1}{\varphi(D^2)}\varphi(-a^2)\sin(ax+\theta)$$
$$\sin(ax+\theta) = \varphi(-a^2)\frac{1}{\varphi(D^2)}\sin(ax+\theta)$$

Now if $\varphi(-a^2) \neq 0$ this can be interpreted as

$$\frac{1}{\varphi(D^2)}\sin(ax+\theta) = \frac{1}{\varphi(-a^2)}\sin(ax+\theta)$$

This beautiful result then states a rule that an n^{th} order Non-Homogeneous Linear DE *with Constant coefficients*

$$L(D^2)y = A\sin(ax+\theta)$$

has the PI

$$y = A\frac{\sin(ax+\theta)}{L(-a^2)}$$

which needs no integration to be performed.

There may arise a situation where $L(-a^2)=0$. This would then imply "$-a^2$" to be an r^{th} *order root* of the *DE so that*

$$L(D^2) = (D^2+a^2)^r\varphi(D^2).$$

The DE can then be written as

$$\varphi(D^2)(D^2+a^2)^r y = A\sin(ax+\theta)$$

Operating on both sides by the "*inverse*" operator $\frac{1}{\varphi(D^2)}$ we find

$$\frac{1}{\varphi(D^2)}\varphi(D^2)(D^2+a^2)^r y = \frac{1}{\varphi(D^2)}A\sin(ax+\theta)$$

$$(D^2+a^2)^r y = A\frac{\sin(ax+\theta)}{\varphi(-a^2)}$$

From our previously learnt technique this yields

$$y = \frac{A}{\varphi(-a^2)}\frac{1}{(D^2+a^2)^r}\sin(ax+\theta)$$

There may also arise a situation wherein the DE contains odd powers of D too. This would then imply

$$(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_4D^4 + a_3D^3 + a_2D^2 + a_1D + a_0)\sin(ax+\theta)$$
$$= [a_nD^n + \cdots + a_4(-a^2)^2 + a_3D^3 + a_2(-a^2) + a_1D + a_0]\sin(ax+\theta)$$

$$L(D)\sin(ax+\theta) = \varphi(D)\sin(ax+\theta)$$

Operating on both sides by the "*inverse*" operator $\frac{1}{\varphi(D^2)}$ we find

$$\frac{1}{\varphi(D^2)}\varphi(D^2)(D^2+a^2)^r y = \frac{1}{\varphi(D^2)}A\sin(ax+\theta)$$

$$(D^2+a^2)^r y = A\frac{\sin(ax+\theta)}{\varphi(-a^2)}$$

From our previously learnt technique this yields

$$y = \frac{A}{\varphi(-a^2)}\frac{1}{(D^2+a^2)^r}\sin(ax+\theta)$$

Example 9.3.1 Solve the equation change

$$\boldsymbol{y'' + 4y = \cos 3x}$$

Solution:
Step 1 The *DE* will be written with the D operator by replacing $y'' \to D^2y$ & $y' \to Dy$

$$(D^2+4)y = \cos 3x$$
$$L(D)y = \cos 3x$$

Step 2 The *Auxiliary Equation* for the corresponding homogeneous *DE*

$$L(D)y = 0$$

will be obtained by writing

$$L(\lambda) = 0$$
$$\lambda^2 + 4 = 0$$
$$\lambda^2 = -4$$

The roots are then found as

$$\lambda_1 = +i2 \ \& \ \lambda_2 = -i2$$

The CF would be $C_1e^{i2x} + C_2e^{-i2x}$ which can be represented as

$$CF = C_1 \cos 2x + C_2 \sin 2x$$

Step 3 The PI would now be obtained as

$$PI = \frac{1}{L(D)} \cos 3x$$

Since $f(x) = \cos(ax)$ is an exponential function we will use the rule $\frac{1}{L(D^2)} \cos(ax) = \frac{1}{L(-a^2)} \cos(ax)$ to find the PI

$$PI = \frac{1}{(D^2+4)} \cos 3x$$
$$PI = \frac{\cos 3x}{\{(-3^2)+4\}}$$
$$PI = \frac{\cos 3x}{\{-9+4\}}$$
$$PI = -\frac{1}{5} \cos 3x$$

Step 4 The *General Solution* would therefore be

$$y = CF + PI = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{5} \cos 3x$$

Example 9.3.2 Solve the equation

$$\boldsymbol{y'' + 2n \cos \alpha \, y' + n^2 y = \sin nx}$$

Solution:
Step 1 The *DE* will be written with the D operator by replacing $y'' \to D^2y$ & $y' \to Dy$

$$(D^2 + 2n\cos\alpha\, D + n^2)y = \sin nx$$
$$L(D)y = \sin nx$$

Step 2 The *Auxiliary Equation* for the corresponding homogeneous *DE*

$$L(D)y = 0$$

will be obtained by writing

$$L(\lambda) = 0$$
$$\lambda^2 + 2n\cos\alpha\,\lambda + n^2 = 0$$

The roots are then found as

$$\lambda_1 = \frac{-(2n\cos\alpha) + \sqrt{(2n\cos\alpha)^2 - 4(n^2)}}{2} \;\&\; \lambda_2 = \frac{-(2n\cos\alpha) - \sqrt{(2n\cos\alpha)^2 - 4(n^2)}}{2}$$
$$\lambda_1 = \frac{-(2n\cos\alpha) + 2n\sqrt{\cos^2\alpha - 1}}{2} \;\&\; \lambda_2 = \frac{-(2n\cos\alpha) - 2n\sqrt{\cos^2\alpha - 1}}{2}$$
$$\lambda_1 = -n\cos\alpha + in\sin\alpha \;\&\; \lambda_2 = -n\cos\alpha - in\sin\alpha$$

The CF would be $C_1e^{(-n\cos\alpha + in\sin\alpha)x} + C_2e^{(-n\cos\alpha - in\sin\alpha)x}$ which can be represented as

$$CF = e^{-(n\cos\alpha)x}\{C_1\cos[(n\sin\alpha)x] + C_2\sin[(n\sin\alpha)x]\}$$

Step 3 The PI would now be obtained as

$$PI = \frac{1}{L(D)}\sin nx = \frac{1}{(D^2 + 2n\cos\alpha\, D + n^2)}\sin nx$$
$$PI = \frac{1}{\left((-n^2) + 2n\cos\alpha\, D + n^2\right)}\sin nx$$
$$PI = \frac{1}{(2n\cos\alpha\, D)}\sin nx$$
$$PI = \frac{1}{(2n\cos\alpha)}\int \sin nx\, dx$$
$$PI = \frac{1}{(2n^2\cos\alpha)}\int \sin nx\, d(nx)$$
$$PI = -\frac{\cos nx}{(2n^2\cos\alpha)}$$

Step 4 The *General Solution* would therefore be

$$y = CF + PI = e^{-(n\cos\alpha)x}\{C_1\cos[(n\sin\alpha)x] + C_2\sin[(n\sin\alpha)x]\} - \frac{\cos nx}{(2n^2\cos\alpha)}$$

9.4 *When function $f(x)$ is of the form x^m, m being a positive integer*

If $f(x) = x^m$ then we can see that

$$Dx^m = mx^{m-1}$$
$$D^2x^m = D(Dx^m) = D(mx^{m-1}) = m(m-1)x^{m-2}$$

and so on

$$D^n x^m = m(m-1)(m-2)\dots(m-n+1)x^{m-n}$$

So if $n = m+1$ then $D^n x^m = 0$ and $D^n x^m = 0\ \forall\ n > m+1$. With this in mind, to evaluate $\frac{1}{L(D)}x^m$ we do the following

- Expand $\frac{1}{L(D)}$ in ascending powers of D as far as the term D^m as we would do for any polynomial expression
- Then operate on x^m by the different powers of D in the expression

Example 9.4.1 Solve the equation

$$\boldsymbol{y'' - 5y' + 6y = x}$$

Solution:

Step 1 The *DE* will be written with the D operator by replacing $y'' \to D^2y$ & $y' \to Dy$

$$(D^2 - 5D + 6)y = x$$
$$L(D)y = x$$

Step 2 The *Auxiliary Equation* for the corresponding homogeneous *DE* $L(D)y = 0$ will be obtained by writing $L(\lambda) = 0$

$$\lambda^2 - 5\lambda + 6 = 0$$

The roots are then found as

$$\lambda_1 = \frac{-(-5) + \sqrt{(-5)^2 - 4(6)}}{2} \ \&\ \lambda_2 = \frac{-(-5) - \sqrt{(-5)^2 - 4(6)}}{2}$$
$$\lambda_1 = \frac{5+1}{2} = 3 \ \&\ \lambda_2 = \frac{5-1}{2} = 2$$

The CF would be

$$CF = C_1e^{3x} + C_2e^{2x}$$

Step 3 The PI would now be obtained as

$$PI = \frac{1}{L(D)}x = \frac{1}{(D^2 - 5D + 6)}x$$
$$PI = \frac{1}{6\left(1 + \frac{(D^2 - 5D)}{6}\right)}x = \frac{1}{6}\left(1 + \frac{(D^2 - 5D)}{6}\right)^{-1} x$$

Since $f(x) = x$ is of power 1 we will expand only upto 1 power of D (any higher power term will vanish as shown earlier)

$$PI = \frac{1}{6}\left(1 + (-1)\frac{(D^2 - 5D)}{6} + \cdots\right)x = \frac{1}{6}\left(1 + \frac{5D}{6}\right)x$$
$$PI = \frac{1}{6}\left(x + \frac{5}{6}\right) = \frac{x}{6} + \frac{5}{36}$$

Step 4 The *General Solution* would therefore be

$$y = CF + PI = C_1 e^{3x} + C_2 e^{2x} + \frac{x}{6} + \frac{5}{36}$$

Example 9.4.2 Solve the equation

$$\boldsymbol{y'' + y' = x^3 + 2x^2}$$

Solution:
Step 1 The *DE* will be written with the D operator by replacing $y'' \to D^2 y$ & $y' \to Dy$

$$(D^2 + D)y = x^3 + 2x^2$$
$$L(D)y = x^3 + 2x^2$$

Step 2 The *Auxiliary Equation* for the corresponding homogeneous *DE* $L(D)y = 0$ will be obtained by writing $L(\lambda) = 0$

$$\lambda^2 + \lambda = 0$$
$$\lambda(\lambda + 1) = 0$$

The roots are then found as

$$\lambda_1 = 0 \,\&\, \lambda_2 = -1$$

The CF would be

$$CF = C_1 e^{0x} + C_2 e^{-x} = C_1 + C_2 e^{-x}$$

Step 3 The PI would now be obtained as

$$PI = \frac{1}{L(D)}(x^3 + 2x^2) = \frac{1}{(D^2 + D)}(x^3 + 2x^2) = \frac{1}{D(D+1)}(x^3 + 2x^2) = \frac{1}{D}(1 + D)^{-1}(x^3 + 2x^2)$$

Since $f(x) = x^3 + 2x^2$ is of power 3 we will expand only upto 3 power of D (any higher power term will vanish as shown earlier)

$$PI = \frac{1}{D}(1 - D + D^2 - D^3 + \cdots)(x^3 + 2x^2) = \frac{1}{D}(1 - D + D^2 - D^3)(x^3 + 2x^2)$$
$$PI = \frac{1}{D}\big((x^3 + 2x^2) - D(x^3 + 2x^2) + D^2(x^3 + 2x^2) - D^3(x^3 + 2x^2)\big)$$
$$PI = \frac{1}{D}\big((x^3 + 2x^2) - (3x^2 + 4x) + (6x + 4) - (6 + 0)\big)$$
$$PI = \frac{1}{D}(x^3 + 2x^2 - 3x^2 + 6x - 4x + 4 - 6) = \frac{1}{D}(x^3 - x^2 + 2x - 2)$$
$$PI = \frac{x^4}{4} - \frac{x^3}{3} + x^2 - 2x$$

Step 4 The *General Solution* would therefore be

$$y = CF + PI = C_1 + C_2 e^{-x} + \frac{x^4}{4} - \frac{x^3}{3} + x^2 - 2x$$

Example 9.4.3 Solve the equation

$$\boldsymbol{y'' + y' - 2y = x + \sin x}$$

Solution:
Step 1 The *DE* will be written with the D operator by replacing $y'' \to D^2 y$ & $y' \to Dy$

$$(D^2 + D - 2)y = x + \sin x$$
$$L(D)y = x + \sin x$$

Step 2 The *Auxiliary Equation* for the corresponding homogeneous *DE* $L(D)y = 0$ will be obtained by writing $L(\lambda) = 0$

$$\lambda^2 + \lambda - 2 = 0$$

The roots are then found as

$$\lambda_1 = \frac{-(1) + \sqrt{(1)^2 - 4(-2)}}{2} = \frac{-1 + \sqrt{9}}{2} \ \& \ \lambda_2 = \frac{-(1) - \sqrt{(1)^2 - 4(-2)}}{2} = \frac{-1 - \sqrt{9}}{2}$$
$$\lambda_1 = 1 \ \& \ \lambda_2 = -2$$

The CF would be

$$CF = C_1 e^x + C_2 e^{-2x}$$

Step 3 The PI would now be obtained as

$$PI = \frac{1}{L(D)}(x + \sin x) = \frac{1}{L(D)}x + \frac{1}{L(D)}\sin x$$
$$PI = \frac{1}{(D^2 + D - 2)}x + \frac{1}{(D^2 + D - 2)}\sin x$$

Let's first solve for

$$PI_1 = \frac{1}{(D^2 + D - 2)}x = -\frac{1}{2\left(1 - \frac{(D^2 + D)}{2}\right)}x = -\frac{1}{2}\left(1 - \frac{(D^2 + D)}{2}\right)^{-1} x$$

Since $f(x) = x$ is of power 1 we will expand only upto 1 power of D (any higher power term will vanish as shown earlier)

$$PI_1 = -\frac{1}{2}\left(1 + \frac{(D^2 + D)}{2}\right)x = -\frac{1}{2}\left(1 + \frac{(D^2 + D)}{2}\right)x = -\frac{1}{2}\left(x + \frac{1}{2}\right)$$

Now let's solve for

$$PI_2 = \frac{1}{(D^2 + D - 2)}\sin x = \frac{1}{\left((-1^2) + D - 2\right)}\sin x = \frac{1}{(D - 3)}\sin x$$

$$PI_2 = \frac{(D + 3)}{(D + 3)(D - 3)}\sin x = \frac{(D + 3)}{(D^2 - 9)}\sin x = \frac{(D + 3)}{\left((-1^2) - 9\right)}\sin x = -\frac{(D + 3)}{10}\sin x$$

$$PI_2 = -\frac{1}{10}(\cos x + 3\sin x)$$

Thus,

$$PI = PI_1 + PI_2 = -\frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(\cos x + 3\sin x)$$

Step 4 The *General Solution* would therefore be

$$y = CF + PI = C_1 e^x + C_2 e^{-2x} - \frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(\cos x + 3\sin x)$$

9.5 *When function $f(x)$ is of the form $e^{ax}V(x)$*

If $f(x) = e^{ax}V(x)$ then we can see that

$$D\{e^{ax}V(x)\} = \{De^{ax}\}\,V(x) + e^{ax}\,\{DV(x)\} = \{ae^{ax}\}\,V(x) + e^{ax}\,\{DV(x)\}$$
$$D\{e^{ax}V(x)\} = e^{ax}\{(D+a)V(x)\}$$

Writing $V_1(x) = (D+a)V(x)$ we find that

$$D\{e^{ax}V(x)\} = e^{ax}V_1(x)$$

Therefore,

$$D^2\{e^{ax}V(x)\} = D\big\{D\{e^{ax}V(x)\}\big\} = D\{e^{ax}V_1(x)\} = e^{ax}\{(D+a)V_1(x)\} = e^{ax}\{(D+a)(D+a)V(x)\}$$
$$D^2\{e^{ax}V(x)\} = e^{ax}\{(D+a)^2V(x)\}$$

This suggests that in general,

$$D^n\{e^{ax}V(x)\} = e^{ax}\{(D+a)^nV(x)\}$$

So if $L(D) = a_nD^n + \cdots + a_2D^2 + a_1D + a_0$ then

$$L(D)\{e^{ax}V(x)\} = (a_nD^n + \cdots + a_2D^2 + a_1D + a_0)\{e^{ax}V(x)\}$$
$$L(D)\{e^{ax}V(x)\} = a_nD^n\{e^{ax}V(x)\} + \cdots + a_2D^2\{e^{ax}V(x)\} + a_1D\{e^{ax}V(x)\} + a_0\{e^{ax}V(x)\}$$
$$L(D)\{e^{ax}V(x)\} = a_ne^{ax}\{(D+a)^nV(x)\} + \cdots + a_2e^{ax}\{(D+a)^2V(x)\} + a_1e^{ax}\{(D+a)V(x)\} + a_0\{e^{ax}V(x)\}$$
$$L(D)\{e^{ax}V(x)\} = e^{ax}[a_n(D+a)^n + \cdots + a_2(D+a)^2 + a_1(D+a) + a_0]V(x)$$

$$L(D)\{e^{ax}V(x)\} = e^{ax}L(D+a)V(x)$$

Thus, operating on both sides by the "*inverse*" operator $\frac{1}{L(D)}$ we find that

$$\frac{1}{L(D)}L(D)\{e^{ax}V(x)\} = \frac{1}{L(D)}\{e^{ax}L(D+a)V(x)\}$$
$$e^{ax}V(x) = \frac{1}{L(D)}\{e^{ax}L(D+a)V(x)\}$$

Now if we write $U(x) = L(D+a)V(x)$ then this can be interpreted as

$$e^{ax}\left\{\frac{1}{L(D+a)}U(x)\right\} = \frac{1}{L(D)}\{e^{ax}U(x)\}$$

This beautiful result then states a rule that an n^{th} order Non-Homogeneous Linear DE with Constant coefficients $L(D)y = e^{ax}V(x)$ has the PI

$$y = \frac{1}{L(D)}\{e^{ax}V(x)\} = e^{ax}\left\{\frac{1}{L(D+a)}V(x)\right\}$$

which simplifies the procedure by taking out the exponential term and displacing the D operatorin $L(D)$ by 'a'.

Example 9.5.1 Solve the equation

$$\boldsymbol{y'' - 2y' + 5y = e^{2x}\sin x}$$

Solution:

Step 1 The *DE* will be written with the D operator by replacing $y'' \to D^2y$ & $y' \to Dy$

$$(D^2 - 2D + 5)y = e^{2x}\sin x$$
$$L(D)y = e^{2x}\sin x$$

Step 2 The *Auxiliary Equation* for the corresponding homogeneous *DE* $L(D)y = 0$ will be obtained by writing $L(\lambda) = 0$

$$\lambda^2 - 2\lambda + 5 = 0$$

The roots are then found as

$$\lambda_1 = \frac{-(-2) + \sqrt{(-2)^2 - 4(5)}}{2} = \frac{2 + \sqrt{-16}}{2} \; \& \; \lambda_2 = \frac{-(-2) + \sqrt{(-2)^2 - 4(5)}}{2} = \frac{2 - \sqrt{-16}}{2}$$
$$\lambda_1 = 1 + i2 \; \& \; \lambda_2 = 1 - i2$$

The CF would be $C_1e^{(1+i2)x} + C_2e^{(1+i2)x}$ which can be represented as

$$CF = e^x\{C_1\cos 2x + C_2\sin 2x\}$$

Step 3 The PI would now be obtained as

$$PI = \frac{1}{L(D)}e^{2x}\sin x = e^{2x}\frac{1}{L(D+2)}\sin x$$
$$PI = e^{2x}\frac{1}{\{(D+2)^2 - 2(D+2) + 5\}}\sin x = e^{2x}\frac{1}{\{D^2 + 4 + 4D - 2D - 4 + 5\}}\sin x$$
$$PI = e^{2x}\frac{1}{\{D^2 + 2D + 5\}}\sin x$$

Now using the rule $\frac{1}{L(D^2)}\{\sin(ax)\} = \frac{1}{L(-a^2)}\{\sin(ax)\}$ we get

$$PI = e^{2x}\frac{1}{\{(-1^2) + 2D + 5\}}\sin x = e^{2x}\frac{1}{\{2D + 4\}}\sin x = \frac{e^{2x}}{2}\frac{1}{(D+2)}\sin x$$
$$PI = \frac{e^{2x}}{2}\frac{(D-2)}{(D-2)(D+2)}\sin x = \frac{e^{2x}}{2}\frac{(D-2)}{(D^2-4)}\sin x = \frac{e^{2x}}{2}\frac{(D-2)}{\left((-1^2) - 4\right)}\sin x$$
$$PI = -\frac{e^{2x}}{10}(D-2)\sin x = -\frac{e^{2x}}{10}(\cos x - 2\sin x)$$
$$PI = \frac{e^{2x}}{10}(2\sin x - \cos x)$$

Step 4 The *General Solution* would therefore be

$$y = CF + PI = e^{x}\{C_1 \cos 2x + C_2 \sin 2x\} + \frac{e^{2x}}{10}(2\sin x - \cos x)$$

Example 9.5.2 Solve the equation

$$\boldsymbol{y'' + \beta^2 y = Ae^{i\alpha x}x}$$

where α & β are constant real numbers.

Solution:
Step 1 The *DE* will be written with the D operator by replacing $y'' \to D^2 y$ & $y' \to Dy$

$$(D^2 + \beta^2)y = Ae^{i\alpha x}x$$
$$L(D)y = Ae^{i\alpha x}x$$

Step 2 The *Auxiliary Equation* for the corresponding homogeneous *DE* $L(D)y = 0$ will be obtained by writing $L(\lambda) = 0$

$$\lambda^2 + \beta^2 = 0$$
$$\lambda = \sqrt{-\beta^2}$$

The roots are then found as

$$\lambda_1 = i\beta \;\&\; \lambda_2 = -i\beta$$

The CF would be $C_1 e^{i\beta x} + C_2 e^{-i\beta x}$ which can be represented as

$$CF = C_1 \cos \beta x + C_2 \sin \beta x$$

Step 3 The PI would now be obtained as

$$PI = \frac{1}{L(D)}Ae^{i\alpha x}x = Ae^{i\alpha x}\frac{1}{L(D + i\alpha)}x$$
$$PI = Ae^{i\alpha x}\frac{1}{\{(D + i\alpha)^2 + \beta^2\}}x = Ae^{i\alpha x}\frac{1}{\{D^2 + 2i\alpha D - \alpha^2 + \beta^2\}}x$$
$$PI = Ae^{i\alpha x}\frac{1}{\{D^2 + 2i\alpha D + (\beta^2 - \alpha^2)\}}x$$

Since $f(x) = x$ is of power 1 we will expand only upto 1 power of D (any higher power term will vanish as shown earlier)

$$PI = \frac{A}{(\beta^2 - \alpha^2)}e^{i\alpha x}\frac{1}{\left\{1 + \frac{2i\alpha D + D^2}{(\beta^2 - \alpha^2)}\right\}}x = \frac{A}{(\beta^2 - \alpha^2)}e^{i\alpha x}\left\{1 + \frac{2i\alpha D + D^2}{(\beta^2 - \alpha^2)}\right\}^{-1}x$$
$$PI = \frac{A}{(\beta^2 - \alpha^2)}e^{i\alpha x}\left\{1 - \frac{2i\alpha D}{(\beta^2 - \alpha^2)}\right\}x$$
$$PI = \frac{A}{(\beta^2 - \alpha^2)}e^{i\alpha x}\left\{x - \frac{2i\alpha}{(\beta^2 - \alpha^2)}\right\}$$

Step 4 The *General Solution* would therefore be

$$y = CF + PI = C_1 \cos\beta x + C_2 \sin\beta x + \frac{A}{(\beta^2 - \alpha^2)} e^{i\alpha x}\left\{x - \frac{2i\alpha}{(\beta^2 - \alpha^2)}\right\}$$

However if $\alpha = \beta$ then

$$CF = C_1 \cos\alpha x + C_2 \sin\alpha x$$

and form step 3 above

$$PI = Ae^{i\alpha x}\frac{1}{\{D^2 + 2i\alpha D\}}x$$

$$PI = \frac{A}{2i\alpha}e^{i\alpha x}\frac{1}{D}\frac{1}{\left\{1 + \frac{D^2}{2i\alpha D}\right\}}x = \frac{A}{2i\alpha}e^{i\alpha x}\frac{1}{D}\frac{1}{\left\{1 + \frac{D}{2i\alpha}\right\}}x$$

$$PI = \frac{A}{2i\alpha}e^{i\alpha x}\frac{1}{D}\left\{1 + \frac{D}{2i\alpha}\right\}^{-1}x = \frac{A}{2i\alpha}e^{i\alpha x}\frac{1}{D}\left\{1 - \frac{D}{2i\alpha}\right\}x$$

$$PI = \frac{A}{2i\alpha}e^{i\alpha x}\frac{1}{D}\left\{x - \frac{1}{2i\alpha}\right\}$$

$$PI = \frac{A}{2i\alpha}e^{i\alpha x}\int\left\{x - \frac{1}{2i\alpha}\right\}dx$$

$$PI = \frac{A}{2i\alpha}e^{i\alpha x}\left\{\frac{x^2}{2} - \frac{x}{2i\alpha}\right\}$$

$$PI = \frac{A}{4\alpha^2}e^{i\alpha x}\left\{\frac{\alpha x^2}{i} + x\right\}$$

The *General Solution* would therefore be

$$y = CF + PI = C_1 \cos\alpha x + C_2 \sin\alpha x + \frac{A}{4\alpha^2}e^{i\alpha x}\left\{\frac{\alpha x^2}{i} + x\right\}$$

Summary

Particular Integral of Special Forms of the Function $f(x)$

- There are certain special forms of the function $f(x)$ which admits rules for finding PI of the Linear DE with constant coefficients in shorter steps.
- When function $f(x)$ is of the form e^{ax} then the PI $y = A\frac{e^{ax}}{L(a)}$

  There may arise a situation where $L(a) = 0$. This would then imply "a" to be an r^{th} *order root* of the n^{th} *order Non-Homogeneous Linear DE with Constant coefficients so that* $L(D) = (D - a)^r \varphi(D)$ *then the PI*

  $$y = \frac{A}{\varphi(a)}\frac{x^r}{r!}e^{ax}$$

- When function $f(x)$ is of the form $\sin ax$ *or* $\cos ax$ then the PI $y = A\frac{\sin(ax+\theta)}{L(-a^2)}$

  There may arise a situation where $L(-a^2) = 0$. This would then imply "$-a^2$" to be an r^{th} *order root* of the *DE so that* $L(D^2) = (D^2 + a^2)^r \varphi(D^2)$ *then the PI*

  $$y = \frac{A}{\varphi(-a^2)}\frac{1}{(D^2 + a^2)^r}\sin(ax + \theta)$$

- When function $f(x)$ is of the form x^m, m being a positive integer then the *PI* can be found by expanding $\frac{\mathbf{1}}{\boldsymbol{L(D)}}$ in ascending powers of $\boldsymbol{D}$ as far as the term $\boldsymbol{D^m}$ as we would do for any polynomial expression and operating on $\boldsymbol{x^m}$ by the different powers of $\boldsymbol{D}$ in the expression

- When function $\boldsymbol{f}(\boldsymbol{x})$ is of the form $\boldsymbol{e^{ax}V}(\boldsymbol{x})$ then the $PI\ y = e^{ax}\left\{\frac{1}{L(D+a)}V(x)\right\}$

Bibliography/ References / Glossary

1. Advanced Engineering Mathematics by Erwin Kreysig
2. Advanced Engineering Mathematics by Michael D. Greenberg
3. Schaum's Outline: Theory and Problems of Advanced Calculus by Murray R. Spiegel
4. Mathematical Methods in Physical Sciences by Mary L. Boas
5. Calculus & Analytic Geometry by Fobes & Smyth
6. Essential Mathematical Methods by K.F. Riley & M.P. Hobson
7. Schaum's Outline: Theory and Problems of Differential Equations by Richard Bronson
8. Schaum's Outline: Theory and Problems of Differential Equations by Frank Ayres
9. Introductory Course in Differential Equations by Daniel A. Murray
10. Differential Equations by N.M. Kapoor
11. Higher Engineering Mathematics by B S Grewal
12. A Treatise on Differential Equations by A. R. Forsyth