Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Work and energy (I)

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Chapter.5 WORK AND ENERGY(I)

1. **Work and Energy**
2. **Work and kinetic Energy Theorem**
3. **Conservative and non-conservative forces**
4. **Force as a gradient of energy**
5. **Potential energy**
6. **Energy diagram**
7. **Stable and unstable equilibrium**
8. **Summary**
9. **Exercise**

Objectives

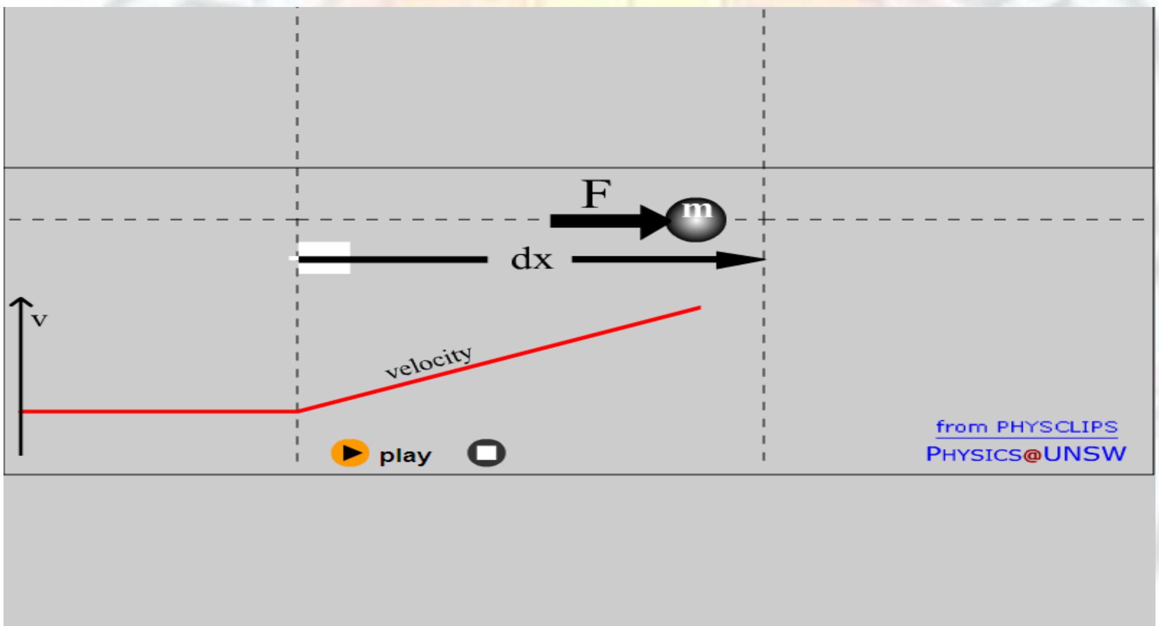
In this chapter you will study:

- Work and energy concept
- The work-Energy theorem
- What are conservative forces and non-conservative forces
- Concept of Potential Energy
- Force representation in term of gradient of the scalar potential
- The Potential Energy versus distance diagram
- Equilibrium and types of equilibrium

1. WORK AND ENERGY

When a force is applied to a particle or the system of particle, the particle is displaced and an amount of work W is done on the particle. This work is done at the cost of energy stored in the particle or the system of particles. So we see that Energy is the capacity of doing work. Now we can define work mathematically as

$$W = \int dW = \int \mathbf{F}.\mathbf{dr}$$



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Where F is the applied force and 'dr' is the infinitesimal displacement of the particle and 'dW' is the infinitesimal work done by the particle.

So we see that work is dot product of force vector and displacement vector, W= **F. r =** f r Cosφ ,

where f and r are the magnitudes of force and displacement vector and φ is the angle between the force and displacement vector.

2. Work and Kinetic Energy Theorem

The work and kinetic energy of a moving body are related to each other this relation is expressed in the form of a theorem as follows:

"The change in the kinetic energy of a particle form initial position to finial position is equal to the work done by the force in displacing the particle from initial position to the finial position."

So, we can write

$$W = K_f - K_i = \frac{1}{2}(mV_f^2\ mV_i^2)$$

Where K_f is the kinetic energy at the finial position, K_i is initial kinetic energy, V_f is the magnitude of final velocity and V_i is initial velocity of the particle of mass m.
Let us prove this theorem.
Suppose a particle of mass m is under an action of a constant and uniform force **F** and let **dr** be the displacement of the particle. Now

$$\mathbf{F} = m\mathbf{a} = m\, d\mathbf{V}/dt$$

also

$$\mathbf{V} = d\mathbf{r}/dt$$

Now work done by the particle in moving from the initial position $\mathbf{r_i}$ to finial position $\mathbf{r_f}$ is given by

$$W = \int_{r_i}^{r_f} \boldsymbol{F}.\boldsymbol{dr}$$

$$W = \int_{r_i}^{r_f} m\boldsymbol{a}.\boldsymbol{dr}$$

$$W = \int_{r_i}^{r_f} m\ \frac{\boldsymbol{dv}}{\boldsymbol{dt}}.\boldsymbol{dr}$$

$$W = \int_{r_i}^{r_f} m\boldsymbol{dv}.\frac{\boldsymbol{dr}}{\boldsymbol{dt}}$$

$$W = \int_{v_i}^{v_f} \boldsymbol{mv}.\boldsymbol{dv}$$

$$W = \frac{1}{2}m(V_f^2 - V_i^2)$$

$$W = K_f - K_i$$

W = ΔK, hence proved.

3. Conservative and non-conservative force

The conservative forces are those forces for which work done is independent of path taken, but it depends only on the initial and finial position. So, we can say that for a closed path the total work done is zero for A CONSERVATIVE FORCE, for the whole journey. Those forces, which are path dependent, are called NON-CONSERVATIVE FORCE. The examples of conservative forces are electrical forces, magnetic forces and gravitational forces. The example of non- conservative forces is dissipative forces like friction and viscous force in a fluid.

Let us study only conservative force in this chapter. Now from the definition of work, as defined in earlier section in this chapter.

We have,

$$W = \int_{r_i}^{r_f} \mathbf{F}.\mathbf{dr} = -\left(\mathrm{V}(\boldsymbol{r_f}) - \mathrm{V}(\boldsymbol{r_i})\right) = \mathrm{V}(\boldsymbol{r_i}) - \mathrm{V}(\boldsymbol{r_f}),$$

where $V(\mathbf{r_i})$ and $V(\mathbf{r_f})$ are some scalar functions which depend upon only at the position. As we shall shortly see that these scalars represent the change in potential energy hence the negative sign. So we see that for conservative forces the work done by the force or the line integral of the force is independent of the path taken and can be written as the difference of some scalar function which depends only the position of initial and finial points of the path. We define this scalar function as the *potential*, **since** it gives us the potential energy, U of the particle, so

$$U = V(\mathbf{r})$$

We can have the equivalent condition for a conservative force, using the vector Identity

$$\text{Curl (grad } \emptyset) = \nabla \times (\nabla\emptyset) = 0 \text{ ,}$$

where ∅ is any scalar function. So using this Identity, we have the necessary and sufficient condition for a conservative force

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = 0,$$

so a force field is conservative if and only if its curl is zero.

Now we know every central force can be expressed as

$$\mathbf{F} = f(\mathbf{r})\hat{\mathbf{r}}$$

So

Curl **F** = $\nabla \times \mathbf{F}$ = $\nabla \times f(\mathbf{r})\hat{\mathbf{r}}$ = $f(\mathbf{r})\,\nabla \times \hat{\mathbf{r}}$ **= 0**

Since the Del operator and unit vector $\hat{\mathbf{r}}$ are parallel vector, so their vector product is zero. Hence we can say that *ALL CENTRAL FORCES ARE CONSERVATIVE FORCES.*

Now, from the identity

Curl (grad ∅) **=** $\nabla \times (\nabla\emptyset)$ =0 ,

We can define conservative force in term of gradient of any scalar function V, called potential, such that **F** = - ∇V, so that

Curl **F** = $\nabla \times \mathbf{F}$ = Curl (grad V(**r**)) = $\nabla \times (\nabla V)$**=**0

4. Force as a gradient of energy

Now the relation between conservative force F and the scalar potential V can be rewritten as **F = -** ∇V

Proof:

Since potential is defined as the line integral of conservative force, so

V(**r**) = $-\int \mathbf{F}.\mathbf{dr}$

Expressing **F** and **r** in terms of their components in three dimensions along the x, y, z axes as

F =F_X **i +** F_Y **j** + F_z **k** **r=** x **i** + y **j** + z**k** and **dr** = dx **i+** dy **j** + dz **k**

So that

$$V(\mathbf{r}) = -\int_{\infty}^{r} (F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}).(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= -\int_{\infty}^{r} (F_x xdx + F_y ydy + F_z zdz)$$

Which on partial differentiation with respect to x, y and z, gives

$F_x = -\frac{\partial V}{\partial x}$ $F_y = -\frac{\partial V}{\partial y}$ $F_z = -\frac{\partial V}{\partial z}$

So $\mathbf{F} = F_X\,\mathbf{i} + F_Y\,\mathbf{j} + F_z\,\mathbf{k}$

$$= -\left[\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right]$$

$$= -\,\text{grad } V$$

Or $\mathbf{F} = -\nabla V$

5. Potential energy

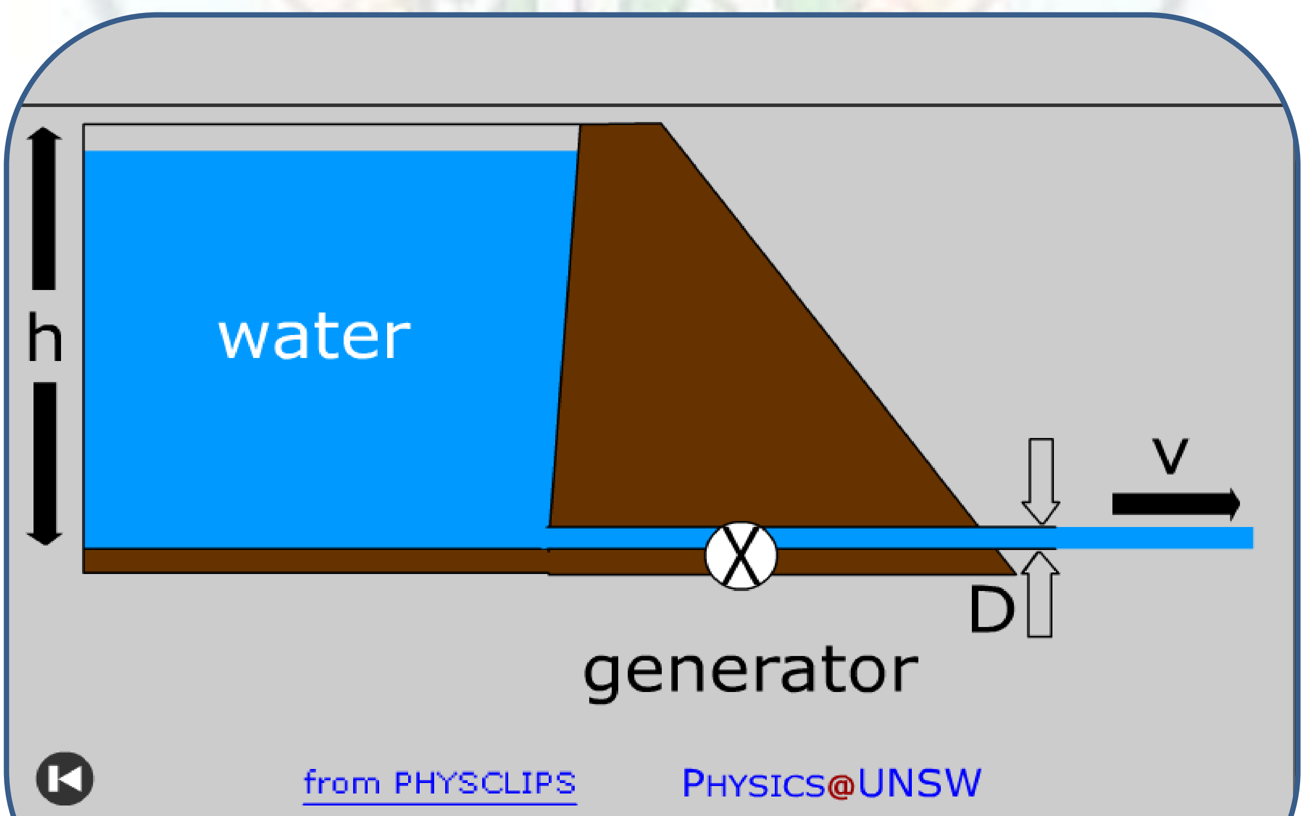
We have already defined the potential energy of a particle or the system of particle as its capacity to do work by virtue of its position. It is measured by the amount of work done by the force to restore the particle from its present position to a fixed position and is denoted by symbol U or V(**r**).

Now we define potential energy as the energy stored in the particle or the system of particles, due to some external force field in the space as,

$$U=W=-\int \mathbf{F.dr}$$

The following multimedia shows an example of the potential energy stored in water in a dam. This stored potential energy can be used to convert mechanical Energy into electrical energy, as shown.

The following example has been taken from http://www.phys.unsw.edu.au/



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Example: The hydroelectric dam problem

The water level in a hydroelectric dam is 100 m above the height at which water comes out of the pipes. Assuming that the turbines and generators are 100% efficient, and neglecting viscosity and turbulence, calculate the flow of water required to produce 10 MW of power. The output pipes have a cross section of 5 m^2.

Solution : This problem has the work-energy theorem, uses power, and requires a bit of thought. Let's do it. Let's consider what is happening in steady state for this system.

Over a time dt, some water of mass dm exits the lower pipe at speed v. This water is delivered to the top of the dam at negligible speed. So the net effect is to take dm of stationary water at height h and deliver it at the bottom of the dam at height zero and speed v.

Let the flow be dm/dt. The work done by the water, dW, is minus the energy increase of the water, so

$$dW = - dE = - dK - dU$$

$$= - (½dm.v^2 - 0) - (0 - dm.gh) = dm(gh - ½v^2).$$

The power delivered is just P = dW/dt.

So

$$\mathbf{P = (gh - ½v^2)dm/dt}$$

Of course the flow dm/dt depends on v.

Let's see how: In time dt, the water flows a distance vdt along the pipe. The cross section of the pipe is A, so the volume of water that has passed a given point is dV = A(vdt).

Using the definition of density,

$\rho = dm/dV$,

we have

$dm/dt = \rho dV/dt = \rho A.(vdt)/dt = \rho Av$.

Substituting in the equation above gives us

$P = \rho Av(gh - \frac{1}{2}v^2)$

Or

$\frac{1}{2}v^3 - ghv + P/\rho A = 0$.

However you look at it, it's a cubic equation, which sounds like a messy solution.

However, let's think of what the terms mean.

The first one came from the kinetic energy term. The second is the work done by gravity. The third is the work done on the turbines. Now, if I had designed this dam, I'd have wanted to convert as much gravitational potential energy as possible into work done on the turbines, so I'd make the pipes wide enough so that the kinetic energy lost by the water outflow would be negligible. Let's see if my guess is correct.

If the first term is negligible, then we simply have $hgv = P/\rho A$.

So $v = P/\rho ghA = 2\ m.s^{-1}$. So the first term would be $4\ m^3.s^{-3}$,

the second would be $- 2000\ m^3.s^{-3}$, and the third would be $2000\ m^3.s^{-3}$.

So yes, the guess was correct and, to the precision required of this problem, the answer is

$v = 2\ m.s^{-1}$.

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6. Energy diagram

Let us consider motion of a particle in an arbitrary potential field of a conservative force. For familiarity, we consider a central force field, like gravitational field. For this, we have equation of motion in one dimension for the particle as

$$ma_r = m\,r'' = F(r) + L^2/mr^3$$

Where r'' is the radial component of acceleration of the particle, m is the mass, F is the magnitude of the force field, L is the angular momentum of the particle. Here L^2/mr^3 is known as centrifugal force. Now, we define effective potential energy V_e as

$$V_e = -\int [F(r) + L^2/mr^3]\,.\,dr$$

Or

$$V_e = V(r) + L^2/2mr^2$$

Here we have used

$$F(r) = -\,dV(r)/dr$$

Now the total energy E of the particle in the central force field is the sum of its kinetic energy and the effective potential energy V_e , Thus

$$E = \tfrac{1}{2} mv^2 + V_e = \tfrac{1}{2} m\,\dot{r}^2 + V_e$$

Since the system is conservative

$$E = T + V = = \tfrac{1}{2} m\,\dot{r}^2 + V_e = \text{constant}$$

So

$$\mathbf{v} = \dot{r} = \sqrt{\left[\frac{2(E - V_e)}{m}\right]}$$

Let us suppose that the particle has total energy E as shown by the dotted line in figure 6.1, the arbitrary potential field is represented as continuous curve. So it is clear from the fig that at points r=r1 and r2, straight line E=constant intersect the potential energy curve.

These points corresponds to $E=V_e$. Hence from the above equation, we have

$$\mathbf{v} = \dot{r} = 0$$

Such points, where the radial components of velocity is zero, are called as turning points. At all other points between r1 and r2 there exist certain differences between the values of E and V_e, this is represented by the ordinate (y-axis) between straight line E= constant and the curve representing V. This is clearly the kinetic energy of the particle.

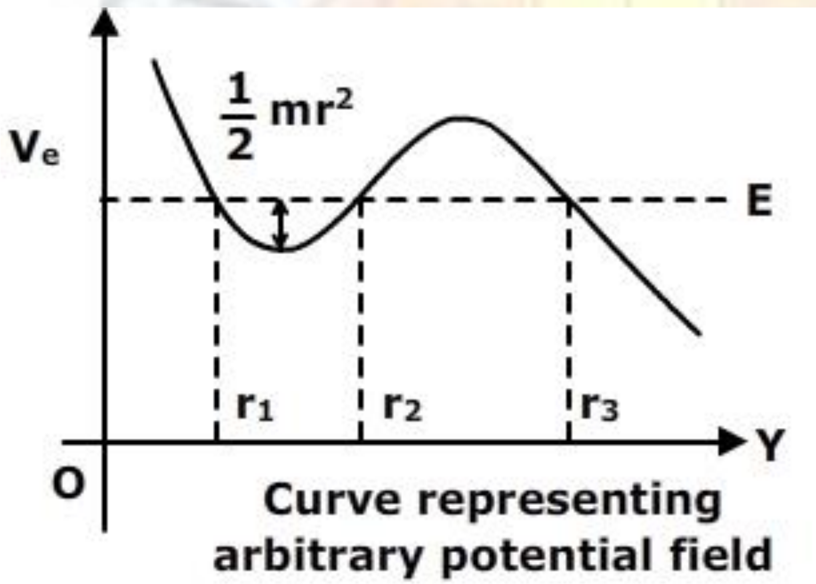

Fig 5.1 Energy diagram for an arbitrary potential field.

7. Stable and unstable equilibrium

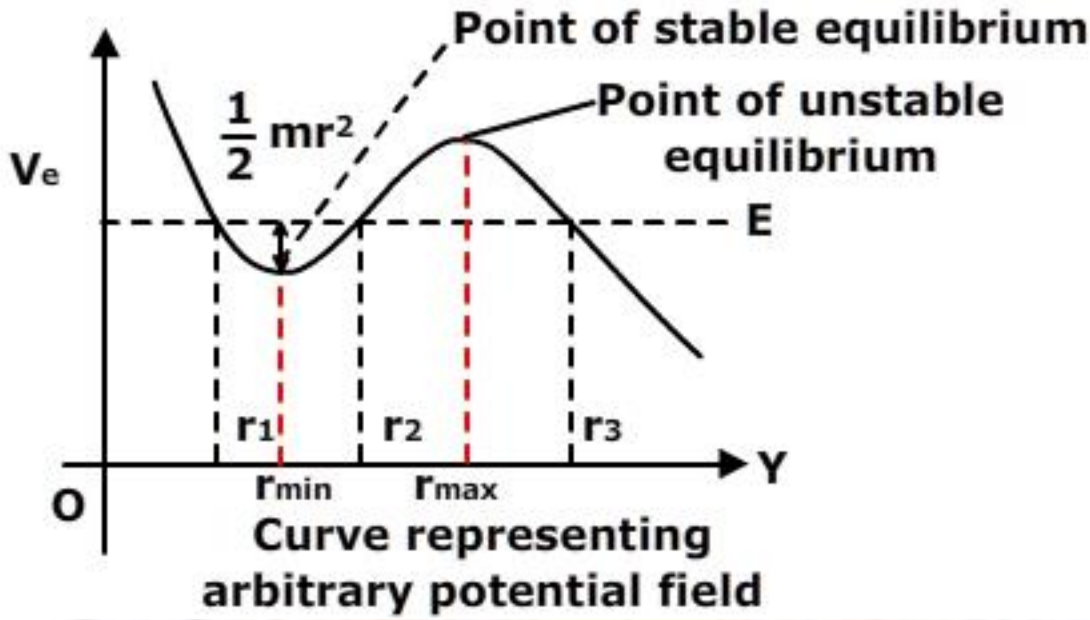

Fig.5.2 Stable and Unstable Equilibrium

Now from the figure 5.2 of the arbitrary potential field, for the whole range of r, we can have different regions, as follows:

1. <u>Region for which r <r1:</u> In this region, the potential energy V_e is greater than the total energy E. hence the kinetic energy will be negative and the velocity will be imaginary. Hence this region is forbidden for the particle.

2. <u>Region for which r1≤r≤r2</u>: In this region, the total energy is greater than the potential energy V_e. This region has r=r1 and r=r2 as the turning points. The motion of particle is therefore, oscillatory in the potential well and the particle will be confined to this region. The particle does not possess enough kinetic energy to cross the potential barriers at r1 and r2. The orbit of the particle may not be closed but
   a. Bounded in this region between two circles of radii r1 and r2. So here the total energy is sum of the kinetic energy and potential energy of the particle.
   b. At r1 and r2 we have kinetic energy zero and the potential energy maximum, and at $\mathbf{r_{min}}$, potential energy is minimum and kinetic energy maximum, so the particle should be at least energy configuration at the point $\mathbf{r_{min}}$, such point is called point of **STABLE EQUILIBRIUM**, where if we displace the particle from this position, it will try to come back to this position again ,that is why it is called stable configuration for the particle.

3. <u>Region for which r2<r<r3;</u> In this region, we again have total energy less than the potential energy, so again kinetic energy is negative and velocity is imaginary. Also potential energy is maximum at the point $\mathbf{r_{max}}$, as shown in the figure. So, if we displace the particle from this position it will never return to this position again, hence this position is of unstable energy configuration for the particle, hence we call

this point as the position of **UNSTABLE EQUILIBRIUM**. Again this region is forbidden for the particle.

8. Summary

- Energy is the capacity of doing work. Now we can define work mathematically as

  $W= \int dW = \int \mathbf{F.dr}$

- The work and kinetic energy of a moving body are related to each other this relation is expressed in the form of a theorem as follows:
  "The change in the kinetic energy of a particle form initial position to finial position is equal to the work done by the force in displacing the particle from initial position to the finial position."

- The conservative forces are those forces for which work done is independent of path taken, but it depends only on the initial and finial position.

- For Conservative Force, we can define a scalar potential V(**r**) such that

  $\mathbf{F} = -\nabla V$ and $V(\mathbf{r}) = -\int \mathbf{F.dr}$

- STABLE EQUILIBRIUM: It is that position in the energy diagram of the potential field of a particle where if we displace the particle from this position ,it will try to come back to this position again .

- UNSTABLE EQUILIBRIUM: If we displace the particle from this position it will never return to this position again, hence this position is of unstable energy configuration for the particle.

9. Exercise

1. Calculate the kinetic energy of a ball of mass 30g moving with the speed of 10m/s.

2. A car of mass 1000kg is moving with the constant speed of 60km/hr, when a man 600m ahead is seen by the driver. If the driver applies his brake so that the car just hit the man with the speed of 1m/s, find the deceleration, time taken and the kinetic energy loss of the car.

3. A 50-kg skydiver moving at terminal speed falls 40 m in 1 second. What power is the skydiver expending on the air? Also find the kinetic energy.

4. How many joules of work are done when a force of 10 N moves a book 5 m?

5. Which requires more work–lifting a 150-kg bucket at a vertical distance of12 m or lifting a 105-kg bucket a vertical distance of24 m?

6.If both buckets in the preceding question are lifted their respective distances in the same time, how does the power required for each compare? How about for the case where the lighter bucket is moved its distance in half the time?

7. How many watts of power are expended when a force of 11 N moves a book 12 m in a time interval of 4 second?

8. Does the force field **F=** yz**i** **-**xz **j+** xy **k** is conservative or non-conservative.

9. A particle have position vector given by **r=** 2x **i +** 3y **j +**4z **k** calculate the work done by the particle, if the applied force is **F=** yz**i** **-**xz **j+** xy **k.**

Fill in the blanks:

10. The capacity to do work is called ____________________.

11. Gravitational and Electric Forces are ______________________forces.

12. The potential energy and Kinetic energy are Collectively known as the Total ___________ Energy.

13. The potential energy vs distance diagrams are known as ____________.

14. The rate of change of the energy w.r.t. time is called ___________________.

State whether the following statements are true or false

15. Work is the vector product of force and displacement.

16. The potential energy curves are known as energy diagrams.

17. Conservative forces are path-dependent forces.

18. Curl of a conservative force is always zero.

19. The minimum of potential energy curve is known as the point of Stable equilibrium.

20. Change in Kinetic energy of a system is equal to the work done by the system.

Choose the most appropriate option for the following:

21. The point of unstable equilibrium is the

(A) Point of minima of the potential energy curve.

(B) Point of maxima of the potential energy curve.

(C) Point of zero slope of the potential energy curve.

22. The potential energy of gravitational field is given by

(A) mgh

(B) mv^2

(C) ma

23. The power delivered by a system is

(A) The rate of change of the linear momentum of the system.

(B) The rate of change of the energy of the system.

(C) The work done by the system.

24. If the curl of a vector field **A** is zero, the vector field is represented as

(A) The dot product of two scalars

(B) The gradient of a scalar function

(C) The vector product of two vector fields.

25. If the total energy is negative for a system, then the motion of the system is

(A) Unbounded

(B) Bounded

(C) Forbidden